FAMILIES OF LOW DIMENSIONAL DETERMINANTAL SCHEMES.

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Abstract. A scheme $X \subset \mathbb{P}^n$ of codimension c is called *standard determinantal* if its homogeneous saturated ideal can be generated by the $t \times t$ minors of a homogeneous $t \times (t+c-1)$ matrix (f_{ij}). Given integers $a_0 \le a_1 \le ... \le a_{t+c-2}$ and $b_1 \le ... \le b_t$, we denote by $W_s(\underline{b}; \underline{a}) \subset \mathrm{Hilb}(\mathbb{P}^n)$ the stratum of standard determinantal schemes where f_{ij} are homogeneous polynomials of degrees $a_j - b_i$ and $\mathrm{Hilb}(\mathbb{P}^n)$ is the Hilbert scheme (if $n - c > 0$, resp. the postulation Hilbert scheme if $n - c = 0$).

Focusing mainly on zero and one dimensional determinantal schemes we determine the codimension of $W_s(\underline{b}; \underline{a})$ in $\mathrm{Hilb}(\mathbb{P}^n)$ and we show that $\mathrm{Hilb}(\mathbb{P}^n)$ is generically smooth along $W_s(\underline{b}; \underline{a})$ under certain conditions. For zero dimensional schemes (only) we find a counterexample to the conjectured value of $\dim W_s(\underline{b}; \underline{a})$ appearing in Kleppe and Miró-Roig [25].

1. Introduction

The goal of this paper is to study maximal families of determinantal schemes. Recall that a scheme $X \subset \mathbb{P}^n$ of codimension c is called determinantal if its homogeneous saturated ideal can be generated by the $r \times r$ minors of a homogeneous $p \times q$ matrix (f_{ij}) with $c = (p-r+1)(q-r+1)$. If $r = \min(p, q)$, then X is called *standard determinantal*. X is called *good determinantal* if it is standard determinantal and a generic complete intersection.

Let $\mathrm{Hilb}(\mathbb{P}^n)$ be the Hilbert scheme (resp. postulation Hilbert scheme, i.e. the Hilbert scheme of constant Hilbert function) parameterizing closed subschemes of $\mathbb{P}^n$ of dimension $n - c > 0$ (resp. $n - c = 0$). Given integers $a_1 \le a_2 \le ... \le a_p$ and $b_1 \le ... \le b_q$, we denote by $W(\underline{b}; \underline{a})$ (resp. $W_s(\underline{b}; \underline{a})$) the stratum in $\mathrm{Hilb}(\mathbb{P}^n)$ consisting of good (resp. standard) determinantal schemes where f_{ij} are homogeneous polynomials of degrees $a_j - b_i$. Then $W_s(\underline{b}; \underline{a})$ is irreducible and $W(\underline{b}; \underline{a}) \neq \emptyset$ if and only if $W_s(\underline{b}; \underline{a}) \neq \emptyset$ (Corollary 2.1).

In this paper we focus, notably for *zero dimensional* schemes, on the following problems.

(1) Determine when the closure of $W(\underline{b}; \underline{a})$ is an irreducible component of $\mathrm{Hilb}(\mathbb{P}^n)$.
(2) Find the codimension of $W(\underline{b}; \underline{a})$ in $\mathrm{Hilb}(\mathbb{P}^n)$ if its closure is not a component.
(3) Determine when $\mathrm{Hilb}(\mathbb{P}^n)$ is generically smooth along $W(\underline{b}; \underline{a})$.

This paper generalizes and completes several results of [24] and [25] for schemes of dimension 0 or 1. Moreover we announced in [25], Rem. 6.3 that [24], §10 contains inaccurate results in the zero dimensional case, which we fully correct in this paper (Remark 4.26).

By successively deleting columns of the matrix associated to a determinantal scheme X, we get a nest ("flag") of closed subschemes $X = X_c \subset X_{c-1} \subset ... \subset X_2 \subset \mathbb{P}^n$. We

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prove our results inductively by considering the smoothness of the Hilbert flag scheme of pairs and its natural projections into the Hilbert schemes. Note that, for $c = 2$, one knows that the closure $\overline{W(\underline{b};\underline{a})}$ is a generically smooth irreducible component of $\mathrm{Hilb}(\mathbb{P}^n)$ (i.e. $\mathrm{Hilb}(\mathbb{P}^n)$ is smooth along some non-empty *open subset* U *of* $\mathrm{Hilb}(\mathbb{P}^n)$ satisfying $U \subset W(\underline{b};\underline{a})$), see Theorem 4.10.

In this approach we need to prove that certain (kernels of) Ext^1-groups vanish or to compute its dimensions. If $\dim X = 1$ (resp. 0), then one (resp. 2 or 3) of these Ext^1-groups may be non-zero and its dimension (resp. the sum of its dimensions) is precisely the codimension of $W(\underline{b};\underline{a})$ in $\mathrm{Hilb}(\mathbb{P}^n)$ under certain assumptions, see Theorem 4.19, Proposition 4.24 and Proposition 4.15 of Section 4. These are main results of this paper, together with Theorem 4.6 which through Proposition 4.13 and Lemma 4.4 give the tools we need in the proofs. As a consequence, if the mentioned Ext^1-groups vanish and $c \le 5$ or 6, we get that the closure $\overline{W(\underline{b};\underline{a})}$ is a generically smooth irreducible component of $\mathrm{Hilb}(\mathbb{P}^n)$ and that every deformation of a general X of $W(\underline{b};\underline{a})$ comes from deforming the defining matrix (f_{ij}) of X. Note that this conclusion holds if $\dim X \ge 2$ and $3 \le c \le 4$ because the above Ext^1-groups vanish by [25] and [24]. If the codimension of $W(\underline{b};\underline{a})$ in $\mathrm{Hilb}(\mathbb{P}^n)$ is positive, there are deformations of X which do not come from deforming the matrix (f_{ij}). In the proofs we use results of [25] and [24] (see Section 3 which also contains a counterexample to the Conjectures of [25] in the case $\dim X = 0$), as well as the Eagon-Northcott and Buchsbaum-Rim complexes ([9],[6], [10]). We give many examples, supported by Macaulay 2 computations [13], to illustrate the results.

As an application we use the results for zero dimensional schemes $X = \mathrm{Proj}(A)$ of this paper, together with the main result of [22] in which artinian Gorenstein rings are obtained by dividing A with ideals being isomorphic to a fixed twist of the canonical module of A, to contribute to the classification of Gorenstein quotients of a polynomial ring of e.g. codimension 4 from the point of view of determining $\mathrm{PGor}(H)$, cf. [19], [21].

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Notation: In this paper $\mathbb{P} := \mathbb{P}^n$ will be the projective n-space over an algebraically closed field k, $R = k[x_0, x_1, \ldots, x_n]$ is a polynomial ring and $\mathfrak{m} = (x_0, \ldots, x_n)$.

We mainly keep the notations of [25]. If $X \subset Y$ are closed subschemes of $\mathbb{P}^n$, we denote by $\mathcal{I}_{X/Y}$ (resp. $\mathcal{N}_{X/Y}$) the ideal (resp. normal) sheaf of X in Y. Note that by the codimension, $\mathrm{codim}_Y X$, of an irreducible X in a not necessarily equidimensional scheme Y we mean $\dim \mathcal{O}_{Y,x} - \dim X$, where x is a general k-point of X. For any closed subscheme X of $\mathbb{P}^n$ of codimension c, we denote by $\mathcal{I}_X$ its ideal sheaf, $\mathcal{N}_X$ its normal sheaf, $I_X = H^0_*(\mathcal{I}_X)$ its saturated homogeneous ideal and we let $\omega_X = \mathcal{E}xt^c_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^n})(-n-1)$. When *we write* $X = \mathrm{Proj}(A)$ *we take* $A := R/I_X$ and $K_A = \mathrm{Ext}^c_R(A, R)(-n-1)$ for the canonical module of A or X. We denote the group of morphisms between coherent $\mathcal{O}_X$-modules by $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ while $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ denotes the sheaf of local morphisms. Moreover we set $\hom(\mathcal{F}, \mathcal{G}) = \dim_k \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ and we correspondingly use small letters for the dimension, as a k-vector space, of similar groups.

We denote the Hilbert scheme by $\mathrm{Hilb}^p(\mathbb{P}^n)$, p the Hilbert polynomial [14], and $(X) \in \mathrm{Hilb}^p(\mathbb{P}^n)$ for the point which corresponds to the subscheme $X \subset \mathbb{P}^n$. We denote by $\mathrm{GradAlg}(H)$, or $\mathrm{Hilb}^H(\mathbb{P}^n)$, the representing object of the functor which parameterizes flat families of graded quotients A of R of $\mathrm{depth}_{\mathfrak{m}} A \geq \min(1, \dim A)$ and with Hilbert function H, $H(i) = \dim A_i$ ([21], [22]), and we call it "the postulation Hilbert scheme" ([23], §1.1) even though it may be different from the parameter space studied by Gotzmann, Iarrobino and others ([12], [19]) who study the "same" scheme with the reduced scheme structure (ours may be non-reduced and is equivalent to the Hilbert scheme of constant postulation considered in [33] in the curve case. They are both special cases of the multigraded Hilbert scheme of Haiman and Sturmfels [17]). Again we let (A), or (X) where $X = \mathrm{Proj}(A)$, denote the point of $\mathrm{GradAlg}(H)$ which corresponds to A. Note that if $\mathrm{depth}_{\mathfrak{m}} A \geq 1$ and ${}_0\mathrm{Hom}_R(I_X, H^1_{\mathfrak{m}}(A)) = 0$, then

$$\mathrm{GradAlg}(H) \simeq \mathrm{Hilb}^p(\mathbb{P}^n) \quad \text{at} \quad (X)\ , \tag{1.1}$$

and hence we have an isomorphism ${}_0\mathrm{Hom}(I_X, A) \simeq H^0(\mathcal{N}_X)$ of their tangent spaces (cf. [11] for the case $\mathrm{depth}_{\mathfrak{m}} A \geq 2$, and [22], (9) for the general case). If (1.1) holds and X is generically a complete intersection, then ${}_0\mathrm{Ext}^1_A(I_X/I_X^2, A)$ is an obstruction space of $\mathrm{GradAlg}(H)$ and hence of $\mathrm{Hilb}^p(\mathbb{P}^n)$ at (X) ([22], §1.1). When we simply write $\mathrm{Hilb}(\mathbb{P}^n)$, we interpret it as the Hilbert scheme (resp. postulation Hilbert scheme) if $n-c > 0$ (resp. $n - c = 0$). By definition X (resp. A) is *unobstructed* if $\mathrm{Hilb}^p(\mathbb{P}^n)$ (resp. $\mathrm{Hilb}^H(\mathbb{P}^n)$) is smooth at (X). Note that we called X *H-unobstructed* in [24] if A was unobstructed.

We say that X is *general* in some irreducible subset $W \subset \mathrm{Hilb}(\mathbb{P}^n)$ if (X) belongs to a sufficiently small open subset U of W such that any (X) in U has all the openness properties that we want to require.

2. Background

In this section we recall some basic results on standard (resp. good) determinantal schemes needed in the sequel, see [3], [4], [10] and [34] for more details. Let

$$\varphi : F = \bigoplus_{i=1}^{t} R(b_i) \longrightarrow G := \bigoplus_{j=0}^{t+c-2} R(a_j) \tag{2.1}$$

be a graded morphism of free R-modules and let $\mathcal{A} = (f_{ij})_{i=1,\dots t}^{j=0,\dots,t+c-2}$, $\deg f_{ij} = a_j - b_i$, be a $t \times (t + c - 1)$ homogeneous matrix which represents the dual $\varphi^* := \mathrm{Hom}_R(\varphi, R)$. Let $I(\mathcal{A}) = I_t(\mathcal{A})$ (or $I_t(\varphi)$) be the ideal of R generated by the maximal minors of $\mathcal{A}$. In the following we always suppose

$$c \geq 2, \quad t \geq 2, \quad b_1 \leq ... \leq b_t \quad \text{and} \quad a_0 \leq a_1 \leq ... \leq a_{t+c-2}. \tag{2.2}$$

Recall that a codimension c subscheme $X \subset \mathbb{P}^n$ is standard determinantal if $I_X = I(\mathcal{A})$ for some homogeneous $t \times (t + c - 1)$ matrix $\mathcal{A}$ as above. Moreover $X \subset \mathbb{P}^n$ is a *good determinantal* scheme if additionally, $\mathcal{A}$ contains a $(t - 1) \times (t + c - 1)$ submatrix (allowing a change of basis if necessary) whose ideal of maximal minors defines a scheme of codimension $c + 1$. Note that if X is standard determinantal and a generic complete intersection in $\mathbb{P}^n$, then X is *good determinantal*, and conversely [27], Thm. 3.4.

Given integers b_i and a_j satisfying (2.2) we let $W(\underline{b};\underline{a})$ (resp. $W_s(\underline{b};\underline{a})$) be the stratum in $\mathrm{Hilb}(\mathbb{P}^n)$ consisting of good (resp. standard) determinantal schemes as above. Since we will not require $\mathcal{A}$ to be minimal (i.e. $f_{ij}=0$ when $b_i=a_j$) for $X=\mathrm{Proj}(R/I_t(\mathcal{A}))$ to belong to $W(\underline{b};\underline{a})$ or $W_s(\underline{b};\underline{a})$, we need to reconsider [25], Cor. 2.6. Indeed looking to its proof and to [25], Rem. 3.7 and the end of p. 2877 (see the Preliminaries of [26] for details), we get

Corollary 2.1. *The closures of* $W(\underline{b};\underline{a})$ *and* $W_s(\underline{b};\underline{a})$ *in* $\mathrm{Hilb}(\mathbb{P}^n)$ *are equal and irreducible. Moreover*

$$W(\underline{b};\underline{a})\neq\emptyset \quad\Leftrightarrow\quad W_s(\underline{b};\underline{a})\neq\emptyset \quad\Leftrightarrow\quad a_{i-1}\geq b_i \ \text{ for all } i \text{ and } a_{i-1}>b_i \ \text{ for some } i\ .$$

Let $A:=R/I_t(\mathcal{A})$ and $M:=\mathrm{coker}(\varphi^*)$. Using the generalized Koszul complexes associated to a codimension c standard determinantal scheme X, one knows, for $\mathcal{A}$ minimal, that the *Eagon-Northcott complex* yields the following minimal free resolution

$$(2.3)\qquad 0\longrightarrow \wedge^{t+c-1}G^*\otimes S_{c-1}(F)\otimes\wedge^tF\longrightarrow \wedge^{t+c-2}G^*\otimes S_{c-2}(F)\otimes\wedge^tF\longrightarrow\dots$$
$$\longrightarrow \wedge^{t}G^*\otimes S_{0}(F)\otimes\wedge^tF\longrightarrow R\longrightarrow A\longrightarrow 0$$

of A and that the *Buchsbaum-Rim complex* yields a minimal free resolution of M;

$$(2.4)\qquad 0\longrightarrow \wedge^{t+c-1}G^*\otimes S_{c-2}(F)\otimes\wedge^tF\longrightarrow \wedge^{t+c-2}G^*\otimes S_{c-3}(F)\otimes\wedge^tF\longrightarrow\dots$$
$$\longrightarrow \wedge^{t+1}G^*\otimes S_{0}(F)\otimes\wedge^tF\longrightarrow G^*\longrightarrow\ F^*\longrightarrow M\longrightarrow 0.$$

See, for instance [4]; Thm. 2.20 and [10]; Cor. A2.12 and Cor. A2.13. Note that (2.3) show that any standard determinantal scheme is arithmetically Cohen-Macaulay (ACM).

Let $\mathcal{B}$ be the matrix obtained deleting the last column of $\mathcal{A}$ and let B be the k-algebra given by the maximal minors of $\mathcal{B}$. Let $Y=\mathrm{Proj}(B)$. The transpose of $\mathcal{B}$ induces a map $\phi: F=\oplus_{i=1}^tR(b_i)\to G':=\oplus_{j=0}^{t+c-3}R(a_j)$. Let $M_{\mathcal{B}}$ be the cokernel of $\phi^*=\mathrm{Hom}_R(\phi,R)$, let $M_{\mathcal{A}}=M$ and suppose $c>2$. In this situation we recall that there is an exact sequence

$$(2.5)\qquad 0\longrightarrow B\longrightarrow M_{\mathcal{B}}(a_{t+c-2})\longrightarrow M_{\mathcal{A}}(a_{t+c-2})\longrightarrow 0$$

in which $B\longrightarrow M_{\mathcal{B}}(a_{t+c-2})$ is a regular section given by the last column of $\mathcal{A}$. Moreover,

$$(2.6)\qquad 0\longrightarrow M_{\mathcal{B}}(a_{t+c-2})^*:=\mathrm{Hom}_B(M_{\mathcal{B}}(a_{t+c-2}),B)\longrightarrow B\longrightarrow A\longrightarrow 0$$

is exact by [27] or [24], (3.1), i.e. we may put $I_{X/Y}:=M_{\mathcal{B}}(a_{t+c-2})^*$. Due to (2.4), M is a maximal Cohen-Macaulay A-module, and so is $I_{X/Y}$ by (2.6). By [10] we have $K_A(n+1)\simeq S_{c-1}M_{\mathcal{A}}(\ell_c)$, and hence $K_B(n+1)\simeq S_{c-2}M_{\mathcal{B}}(\ell_{c-1})$, where

$$(2.7)\qquad \ell_i:=\sum_{j=0}^{t+i-2}a_j-\sum_{k=1}^{t}b_k \ \text{ for }\ 2\leq i\leq c.$$

Hence by successively deleting columns from the right hand side of $\mathcal{A}$, and taking maximal minors, one gets a flag of determinantal subschemes

$$(2.8)\qquad (\mathbf{X}.): X=X_c\subset X_{c-1}\subset\dots\subset X_2\subset\mathbb{P}^n$$

where each $X_{i+1}\subset X_i$ (with ideal sheaf $\mathcal{I}_{X_{i+1}/X_i}=\mathcal{I}_i$) is of codimension 1, $X_i\subset\mathbb{P}^n$ is of codimension i and where there exist $\mathcal{O}_{X_i}$-modules $\mathcal{M}_i$ fitting into short exact sequences

$$(2.9)\qquad 0\to\mathcal{O}_{X_i}(-a_{t+i-1})\to\mathcal{M}_i\to\mathcal{M}_{i+1}\to 0\ \text{ for }\ 2\leq i\leq c-1,$$

such that $\mathcal{I}_i(a_{t+i-1})$ is the $\mathcal{O}_{X_i}$-dual of $\mathcal{M}_i$ for $2 \le i \le c$, and $\mathcal{M}_2$ is a twist of the canonical module of X_2. In this context we let $D_i := R/I_{X_i}$, $I_{D_i} = I_{X_i}$ and $I_i := I_{D_{i+1}/I_{D_i}}$.

Remark 2.2. Let α be a positive integer. If X is general in $W(\underline{b};\underline{a})$ and $a_{i-\min(\alpha,t)} - b_i \ge 0$ for $\min(\alpha,t) \le i \le t$, then X_j, for all $j = 2, \cdots, c$, is non-singular except for a subset of codimension at least $\min\{2\alpha-1, j+2\}$, i.e.

$$\operatorname{codim}_{X_j} Sing(X_j) \ge \min\{2\alpha-1, j+2\}. \tag{2.10}$$

As observed in Rem. 2.7 of [25], this follows from the Theorem of [7] by arguing as in [7], Example 2.1. See also [38], Prop. 1, [1], Sect. 2 and [35] for a related cases. In particular, if $\alpha \ge 3$, we get that for each $i > 0$, the closed embeddings $X_i \subset \mathbb{P}^n$ and $X_{i+1} \subset X_i$ are local complete intersections outside some set Z_i of codimension at least $\min(4, i+1)$ in X_{i+1} ($\operatorname{depth}_{Z_i} \mathcal{O}_{X_{i+1}} \ge \min(4, i+1)$), cf. next paragraph.

In what follows we always let $Z \subset X$ (and similarly for $Z_i \subset X_i$) be some closed subset such that $U := X - Z \hookrightarrow \mathbb{P}^n$ (resp. $U_i := X_i - Z_i \hookrightarrow \mathbb{P}^n$) is a local complete intersection (l.c.i.). Using that the 1^{st} Fitting ideal of M is equal to $I_{t-1}(\varphi)$, we get that $\tilde{M}$ is locally free of rank one precisely on $X - V(I_{t-1}(\varphi))$ [3], Lem. 1.4.8. Since the set of non locally complete intersection points of $X \hookrightarrow \mathbb{P}^n$ is exactly $V(I_{t-1}(\varphi))$ by e.g. [39], Lem. 1.8, we get that $U \subset X - V(I_{t-1}(\varphi))$ and that $\tilde{M}$ is locally free on U. Indeed $\mathcal{M}_i$ and $\mathcal{I}_{X_i}/\mathcal{I}_{X_i}^2$ are locally free on U_i, as well as on $U_{i-1} \cap X_i$. Note that since $V(I_{t-1}(\mathcal{B})) \subset V(I_t(\mathcal{A}))$, we may suppose $Z_{i-1} \subset X_i$!

Finally notice that there is a close relation between $M(a_{t+c-2})$ and the normal module $N_{X/Y}$ of the quotient $B \simeq R/I_Y \to A \simeq R/I_X$. If we suppose $\operatorname{depth}_{I(Z)} B \ge 2$ where now $Y - Z \hookrightarrow \mathbb{P}^n$ is an l.c.i., we get by applying $\operatorname{Hom}_B(I_{X/Y}, -)$ to (2.6), that

$$0 \longrightarrow B \longrightarrow M_{\mathcal{B}}(a_{t+c-2}) \longrightarrow N_{X/Y} \tag{2.11}$$

is exact. Hence we have an injection $M_{\mathcal{A}}(a_{t+c-2}) \hookrightarrow N_{X/Y}$, which in the case $\operatorname{depth}_{I(Z)} B \ge 3$ leads to an isomorphism $M_{\mathcal{A}}(a_{t+c-2}) \simeq N_{X/Y}$. Indeed, this follows from the more general fact (by letting $L = N = I_{X/Y}$) that if L and N are finitely generated B-modules such that $\operatorname{depth}_{I(Z)} L \ge r+1$ and $\tilde{N}$ is locally free on $U := Y - Z$, then the natural map

$$\operatorname{Ext}^i_B(N, L) \longrightarrow H^i_*(U, \mathcal{H}om_{\mathcal{O}_Y}(\tilde{N}, \tilde{L})) \tag{2.12}$$

is an isomorphism (resp. an injection) for $i < r$ (resp. $i = r$), and $H^i_*(U, \mathcal{H}om_{\mathcal{O}_Y}(\tilde{N}, \tilde{L})) \simeq H^{i+1}_{I(Z)}(\operatorname{Hom}_B(N, L))$ for $i > 0$, cf. [15], exp. VI. Note that we interpret $I(Z)$ as $\mathbf{m}$ if $Z = \emptyset$.

3. The dimension of the determinantal locus

In [25] we conjectured the dimension of $W(\underline{b};\underline{a})$ in terms of the invariant

$$\lambda_c := \sum_{i,j} \binom{a_i - b_j + n}{n} + \sum_{i,j} \binom{b_j - a_i + n}{n} - \sum_{i,j} \binom{a_i - a_j + n}{n} - \sum_{i,j} \binom{b_i - b_j + n}{n} + 1. \tag{3.1}$$

Here the indices belonging to a_j (resp. b_i) range over $0 \le j \le t+c-2$ (resp. $1 \le i \le t$), $\binom{a}{n} = 0$ if $a < n$ and we always suppose $W(\underline{b};\underline{a}) \ne \emptyset$ in the following, cf. Corollary 2.1.

Conjecture 3.1. *Given integers* $a_0 \le a_1 \le ... \le a_{t+c-2}$ *and* $b_1 \le ... \le b_t$*, we set* $\ell_i := \sum_{j=0}^{t+i-2} a_j - \sum_{k=1}^{t} b_k$ *and* $h_{i-3} := 2a_{t+i-2} - \ell_i + n$*, for* $i = 3, 4, ..., c$*. Assume* $a_{i-\min([c/2]+1,t)} \ge b_i$ *for* $\min([c/2]+1, t) \le i \le t$*. Then we have*

$$\dim W(\underline{b};\underline{a}) = \lambda_c + K_3 + K_4 + ... + K_c \ , \quad (\dim W(\underline{b};\underline{a}) = \lambda_2 \text{ if } c = 2)$$

where $K_3 = \binom{h_0}{n}$ *and* $K_4 = \sum_{j=0}^{t+1} \binom{h_1+a_j}{n} - \sum_{i=1}^{t} \binom{h_1+b_i}{n}$ *and in general*

$$K_{i+3} = \sum_{\substack{r+s=i \\ r,s\ge 0}} \ \sum_{\substack{0\le i_1<...<i_r\le t+i \\ 1\le j_1\le...\le j_s\le t}} (-1)^{i-r} \binom{h_i + a_{i_1} + \cdots + a_{i_r} + b_{j_1} + \cdots + b_{j_s}}{n} \text{ for } 0 \le i \le c-3.$$

For the special case where all the entries of $\mathcal{A}$ have the same degree, this means:

Conjecture 3.2. *Let* $W(\underline{0};\underline{d})$ *be the locus of good determinantal schemes in* $\mathbb{P}^n$ *of codimension c given by the maximal minors of a* $t\times(t+c-1)$ *matrix with entries homogeneous forms of degree d. Then,*

$$\dim W(\underline{0};\underline{d}) = t(t+c-1)\binom{d+n}{n} - t^2 - (t+c-1)^2 + 1.$$

In [25] we proved that the right hand side in the formula for $\dim W(\underline{b};\underline{a})$ in the Conjectures is always an upper bound for $\dim W(\underline{b};\underline{a})$ ([25], Thm. 3.5), and moreover, that the Conjectures hold in the range

(3.2) $\qquad 2 \le c \le 5 \ \text{ and } \ n - c > 0$ (supposing $char k = 0$ if $c = 5$) ,

as well as for large classes in the range $c \ge 2$ (without assuming $n > c$), cf. [25], §4.

Example 3.3 (Counterexample to the Conjectures in the range $n = c \ge 3$)**.**

Let $\mathcal{A}$ be a general $2\times(c+1)$ matrix of linear entries. The vanishing of all 2×2 minors defines a reduced scheme X of $c+1$ different points in $\mathbb{P}^c$. The conjectured dimension is $c(c+1)+c-2$ while the dimension of the postulation Hilbert scheme, $\dim_{(X)} \mathrm{Hilb}^H(\mathbb{P}^c)$ is at most $c(c+1)$. Hence

$$\dim W(0,0;1,1,...,1) \le c(c+1) \ .$$

This contradicts both conjectures
for every $c \ge 3$.

We have, however, looked at many examples in the range $a_0 > b_t$ where we have used Macaulay 2 to compute necessary invariants (cf. (3.3) below), without finding more counterexamples. The counterexample we have is only for zero dimensional schemes. Mainly because of this example the Conjecture is slightly changed (for zero-schemes) in [26], Conj. 4.1, excluding Example 3.3 from the new conjecture.

Now we recall a few statements from the proof of (3.2) and a variation which we will need in the next section. In the proof we used induction on c by successively deleting columns of the largest possible degree. Hence we computed the dimension of $W(\underline{b};\underline{a})$, $\underline{a} = a_0, a_1, ..., a_{t+c-2}$ in terms of dimension of $W(\underline{b};\underline{a'})$, where $\underline{a'} = a_0, a_1, ..., a_{t+c-3}$. As in §2, we let $X = \mathrm{Proj}(A)$ belong to $W(\underline{b};\underline{a})$ and we let $Y = \mathrm{Proj}(B)$, $(Y) \in W(\underline{b};\underline{a'})$, be obtained by deleting the last column of $\mathcal{A}$. We have

Proposition 3.4. *Let $c \geq 3$, let $(X) \in W(\underline{b};\underline{a})$ and suppose* $\dim W(\underline{b};\underline{a'}) \geq \lambda_{c-1} + K_3 + K_4 + ... + K_{c-1}$ *and* $\operatorname{depth}_{I(Z)} B \geq 2$ *for a general* $Y = \operatorname{Proj}(B) \in W(\underline{b};\underline{a'})$. *If*

$$ {}_0\hom_R(I_Y, I_{X/Y}) \leq \sum_{j=0}^{t+c-3} \binom{a_j - a_{t+c-2} + n}{n} , \tag{3.3} $$

then $\dim W(\underline{b};\underline{a}) = \lambda_c + K_3 + K_4 + ... + K_c$. *We also get equality in* (3.3), *as well as*

$$ \dim W(\underline{b};\underline{a}) = \dim W(\underline{b};\underline{a'}) + \dim_k M_{\mathcal{B}}(a_{t+c-2})_0 - 1 - {}_0\hom_R(I_Y, I_{X/Y}). $$

Proof. Indeed the proof of Thm. 4.5 of [25] contains the ideas we need, but since the assumptions of Thm. 4.5 are different, we include a proof. First we remark that we have

$$ \lambda_c + K_3 + K_4 + ... + K_c \geq \dim W(\underline{b};\underline{a}) $$

by [25], Prop. 3.13 which combined with the assumption on $\dim W(\underline{b};\underline{a'})$ yields

$$ \lambda_c - \lambda_{c-1} - K_c \geq \dim W(\underline{b};\underline{a}) - \dim W(\underline{b};\underline{a'}). $$

Next by [25], Prop. 4.1 we have the inequality

$$ \dim W(\underline{b};\underline{a}) - \dim W(\underline{b};\underline{a'}) \geq \dim_k M_{\mathcal{B}}(a_{t+c-2})_0 - 1 - {}_0\hom_R(I_Y, I_{X/Y}). $$

Since $K_c = {}_0\hom(\operatorname{coker}\varphi, R(a_{t+c-2}))$ by definition (see [25], Prop. 3.12 and (3.14)) we can use (2.4) and (2.5) to get

$$ \dim M_{\mathcal{B}}(a_{t+c-2})_0 - 1 = \dim M_{\mathcal{A}}(a_{t+c-2})_0 = \tag{3.4} $$

$$ \dim F^*(a_{t+c-2})_0 - \dim G^*(a_{t+c-2})_0 + {}_0\hom(\operatorname{coker}\varphi, R(a_{t+c-2})) = $$

$$ = \sum_{i=1}^{t} \binom{a_{t+c-2} - b_i + n}{n} - \sum_{j=0}^{t+c-2} \binom{a_{t+c-2} - a_j + n}{n} + K_c. $$

Now looking at (3.1) and noticing that λ_{c-1} is defined by the analogous expression where a_j (resp b_i) ranges over $0 \leq j \leq t+c-3$ (resp. $1 \leq i \leq t$), it follows after a straightforward computation that

$$ \lambda_c - \lambda_{c-1} = \sum_{i=1}^{t} \binom{a_{t+c-2} - b_i + n}{n} - \sum_{j=0}^{t+c-2} \binom{a_{t+c-2} - a_j + n}{n} - \sum_{j=0}^{t+c-3} \binom{a_j - a_{t+c-2} + n}{n}. $$

Combining with (3.3), we get

$$ \dim M_{\mathcal{B}}(a_{t+c-2})_0 - 1 - {}_0\hom_R(I_Y, I_{X/Y}) \geq \lambda_c - \lambda_{c-1} + K_c . $$

Hence all inequalities of displayed formulas in this proof are equalities and we are done. □

Theorem 3.5. *The Conjectures (and if $c > 2$, the final dimension formula of Proposition 3.4) hold provided*

$$ 2 \leq c \leq 5 \quad \text{and} \quad n - c > 0 \ (\text{supposing } \mathit{char} k = 0 \text{ if } c = 5\,). $$

Indeed this is mainly [25], Thm. 4.5, Cor. 4.7, Cor. 4.10, Cor. 4.14 and [11] ($c = 2$) and [24] ($c = 3$). Moreover since the proofs of [25] also show (3.3), we get the final dimension formula of Proposition 3.4. Moreover we have (valid also for $n = c$ and $\mathit{char} k \neq 0$):

Remark 3.6. Assume $a_0 > b_t$. Then (3.3) for X general, and Conjecture 3.1 hold provided $3 \leq c \leq 5$ (resp. $c > 5$) and $a_{t+c-2} > a_{t-2}$ (resp. $a_{t+3} > a_{t-2}$) by [26], Thm. 3.2.

4. THE CODIMENSION OF THE DETERMINANTAL LOCUS

In this section we consider the problem of when the closure of $W(\underline{b};\underline{a})$ is an irreducible component of $\mathrm{Hilb}(\mathbb{P}^n)$. If it is not a component, we determine its codimension in $\mathrm{Hilb}(\mathbb{P}^n)$ under certain assumptions. We also examine when $\mathrm{Hilb}(\mathbb{P}^n)$ is generically smooth along $W(\underline{b};\underline{a})$. Moreover we have chosen to introduce the notion "every deformation of X comes from deforming $\mathcal{A}$" because it gives the main reason for why $\overline{W(\underline{b};\underline{a})}$ is not always an irreducible component of $\mathrm{Hilb}(\mathbb{P}^n)$.

In the case the determinantal schemes are of dimension zero or one, then $\overline{W(\underline{b};\underline{a})}$ is not necessarily an irreducible component of $\mathrm{Hilb}(\mathbb{P}^n)$, as the following example shows.

Example 4.1 ($\overline{W(\underline{b};\underline{a})}$ not an irreducible component in the range $0 \le n-c \le 1$, $c \ge 3$)**.** Let $\mathcal{B}$ be a general $2\times c$ matrix of linear entries and let $\mathcal{A} = [\mathcal{B}, v]$ where the entries of the column v are general polynomials of the same degree 2. The vanishing all 2×2 minors of $\mathcal{A}$ defines a determinantal scheme X of codimension c in $\mathbb{P}^n$.

(i) Let $n = c$. Then $X = \mathrm{Proj}(A)$ is a reduced scheme of $2c+1$ points in $\mathbb{P}^c$ and with h-vector $(\dim A_i)_{i=0}^{\infty} = (1, c+1, 2c+1, 2c+1, ...)$. It follows that ${}_vH^1_{\mathbf{m}}(A) \simeq H^1(\mathcal{I}_X(v)) = 0$ for $v \ge 2$ and we get ${}_0\mathrm{Hom}_R(I_X, H^1_{\mathbf{m}}(A)) = 0$. By (1.1) the postulation Hilbert scheme is isomorphic to the usual Hilbert scheme at (X), whose dimension is $c(2c+1)$. Moreover since the dimension of $\overline{W(\underline{b};\underline{a})}$ is at most the conjectured value c^2+4c-2, and since

$$c^2 + 4c - 2 < c(2c+1) \quad \text{for every } c \ge 3,$$

it follows that $\overline{W(0,0;1,1,...,1,2)}$ is not an irreducible component of $\mathrm{Hilb}^H(\mathbb{P}^c)$.

(ii) Let $n = c+1$. Then X is a smooth connected curve in $\mathbb{P}^{c+1}$ of degree $d = 2c+1$ and genus $g = c$. Since $\dim \overline{W(\underline{b};\underline{a})}$ is at most the conjectured value, which is c^2+7c+2, and since $\dim_{(X)} \mathrm{Hilb}^p(\mathbb{P}^{c+1})$ is at least $(n+1)d + (n-3)(1-g) = c^2+8c$, it follows that $\overline{W(0,0;1,1,...,1,2)}$ is not an irreducible component of $\mathrm{Hilb}^p(\mathbb{P}^{c+1})$ for every $c \ge 3$.

In what follows we briefly say "T a local ring" (resp. "T artinian") for a local k-algebra $(T, \mathfrak{m}_T)$ essentially of finite type over $k = T/\mathfrak{m}_T$ (resp. such that $\mathfrak{m}_T^r = 0$ for some integer r). Moreover we say "$T \to S$ is a small artinian surjection" provided there is a morphism $(T, \mathfrak{m}_T) \to (S, \mathfrak{m}_S)$ of local artinian k-algebras whose kernel $\mathfrak{a}$ satisfies $\mathfrak{a}\cdot\mathfrak{m}_T = 0$.

Let $A = R/I_t(\mathcal{A})$. If T is a local ring, we denote by $\mathcal{A}_T = (f_{ij,T})$ a matrix of homogeneous polynomials belonging to the graded polynomial algebra $R_T := R\otimes_k T$, satisfying $f_{ij,T}\otimes_T k = f_{ij}$ and $\deg f_{ij,T} = a_j - b_i$ for all i, j. Note that all elements from T are considered to be of degree zero.

Once having such a matrix $\mathcal{A}_T$, we get an induced morphism

$$\varphi_T : F_T := \oplus_{i=1}^t R_T(b_i) \to G_T := \oplus_{j=0}^{t+c-2} R_T(a_j) \tag{4.1}$$

and we put $M_T = \mathrm{coker}\,\varphi_T^*$.

Lemma 4.2. *If $X = \mathrm{Proj}(A)$, $A = R/I_t(\mathcal{A})$, is a standard determinantal scheme, then $A_T := R_T/I_t(\mathcal{A}_T)$ and M_T are (flat) graded deformations of A and M respectively for every choice of $\mathcal{A}_T$ as above. In particular $X_T = \mathrm{Proj}(A_T) \subset \mathbb{P}^n_T := \mathrm{Proj}(R_T)$ is a deformation of $X \subset \mathbb{P}^n$ to T with constant Hilbert function.*

Proof. Consulting (2.3) and (2.4) we see that the Eagon-Northcott and Buchsbaum-Rim complexes are functorial in the sense that, over R_T, all free modules and all morphisms in these complexes are induced by φ_T, i.e. they are determined by $\mathcal{A}_T$. Since these complexes become free resolutions of A and M respectively when we tensor with k over T, it follows that A_T and M_T are flat over T and satisfy $A_T \otimes_T k = A$ and $M_T \otimes_T k = M$. □

Definition 4.3. Let $X = \mathrm{Proj}(A)$, $A = R/I_t(\mathcal{A})$, be a standard determinantal scheme. We say "every deformation of X comes from deforming $\mathcal{A}$" if for every local ring T and every graded deformation $R_T \to A_T$ of $R \to A$ to T, then A_T is of the form $A_T = R_T/I_t(\mathcal{A}_T)$ for some $\mathcal{A}_T$ as above. Note that by (1.1) we can in this definition replace "graded deformations of $R \to A$" by "deformations of $X \hookrightarrow \mathbb{P}^n$" provided $\dim X \geq 1$.

Lemma 4.4. *Let* $X = \mathrm{Proj}(A)$*,* $A = R/I_t(\mathcal{A})$*, be a standard determinantal scheme,* $(X) \in W(\underline{b}; \underline{a})$*. If every deformation of* X *comes from deforming* $\mathcal{A}$*, then* A *(and hence* X *if* $\dim X \geq 1$*) is unobstructed. Moreover* $\overline{W(\underline{b}; \underline{a})}$ *is an irreducible component of* $\mathrm{Hilb}(\mathbb{P}^n)$.

Proof. Let $T \to S$ be a small artinian surjection and let A_S be a deformation of A to S. By assumption, $A_S = R_S/I_t(\mathcal{A}_S)$ for some matrix $\mathcal{A}_S$. Since $T \to S$ is surjective, we can lift each $f_{ij,S}$ to a polynomial $f_{ij,T}$ with coefficients in T such that $f_{ij,T} \otimes_T S = f_{ij,S}$. By Lemma 4.2 it follows that $A_T := R_T/I_t(\mathcal{A}_T)$ is flat over T. Since $A_T \otimes_T S = A_S$ we get the unobstructedness of A, as well as the unobstructedness of X in the case $\dim X \geq 1$ by (1.1).

Finally let T be the local ring of $\mathrm{Hilb}(\mathbb{P}^n)$ at (X) and let A_T, or $\mathrm{Proj}(A_T)$ if $\dim X \geq 1$, be the pullback of the universal object of $\mathrm{Hilb}(\mathbb{P}^n)$ to $\mathrm{Spec}(T)$. Then there is a matrix $\mathcal{A}_T = (f_{ij,T})$ such that $A_T = R_T/I_t(\mathcal{A}_T)$ by assumption. We can extend $f_{ij,T}$ to polynomials $f_{ij,D}$ with coefficients in D where $\mathrm{Spec}(D) \subset \mathrm{Hilb}(\mathbb{P}^n)$ is an open neighborhood of (X) for which the Eagon-Northcott complex associated to the matrix $\mathcal{A}_D = (f_{ij,D})$ is exact at any $(X') \in \mathrm{Spec}(D)$ (cf. [32], Lem. 6.3 or [11], proof of Thm. 1; in our case the existence of $\mathrm{Spec}(D)$ is quite easy since the Eagon-Northcott complex of the homogeneous coordinate ring of a standard determinantal scheme is always exact). It follows that $\mathrm{Spec}(D) \subset W(\underline{b}; \underline{a})$, and since $\mathrm{Spec}(D)$ is open in $\mathrm{Hilb}(\mathbb{P}^n)$ we are done. □

Remark 4.5. The arguments of these lemmas, which rely on the fact that we get T-flat schemes by just parameterizing the polynomials of $\mathcal{A}$ over a local ring T, is mostly known, see e.g. Laksov's papers [29], [28] where he looks to flat families of determinantal schemes and their singular loci for arbitrary determinantal schemes. Indeed we may expect from Laksov's papers (or prove by other arguments, as we remember Laksov did in a talk at the university of Oslo in the 70's) that families of *arbitrary* determinantal schemes obtained by parameterizing polynomials as above are T-flat; thus he mainly shows the unobstructedness part of Lemma 4.4. The unobstructedness may also easily be deduced from of Schaps' paper [36], Remark to Prop. 1. Since we in this paper only look at determinantal schemes defined through *maximal* minors, our proof only uses the exactness of the Eagon-Northcott complex which somehow contains the argument about generators and relations indicated in [36] as a main ingredient. Surprisingly enough the corresponding unobstructedness result of the *R-module* M (of maximal grade [5]) seems less known. Indeed one may prove the unobstructedness of M as we did for A in Lemma 4.4 because we from the Buchsbaum-Rim complex may see that every deformation of M to

T comes from deforming $\mathcal{A}$. We have learned, by distributing a preliminary version of this paper to specialists in deformations of modules, that the unobstructedness of M was proved in Runar Ile's PhD thesis, cf. [20], ch. 6 (Lem. 6.1.2 or Cor. 6.1.4).

The following result is a key result to our work in this section. Here the morphisms of the Ext^1-groups are induced by the inclusion $I_{X/Y} \hookrightarrow B$, e.g.

$$\tau_{X/Y} \ : \ {}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, I_{X/Y}) \to {}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, B).$$

Theorem 4.6. *Let $X = \mathrm{Proj}(A) \subset Y = \mathrm{Proj}(B)$ be good determinantal schemes defined by the vanishing of the maximal minors of $\mathcal{A}$ and $\mathcal{B}$ respectively where $\mathcal{B}$ is obtained by deleting the last column of $\mathcal{A}$. Let $Z \subset Y$ be a closed subset such that $U := Y - Z \hookrightarrow \mathbb{P}^n$ is a local complete intersection and suppose*

(1) $\mathrm{depth}_{I(Z)} B \geq 3$, <u>or</u>
$\mathrm{depth}_{I(Z)} B \geq 2$ *and* $\rho^1 : {}_0\mathrm{Ext}^1_B(I_{X/Y}, I_{X/Y}) \to {}_0\mathrm{Ext}^1_B(I_{X/Y}, B)$ *is injective,*
(2) $\tau_{X/Y} : {}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, I_{X/Y}) \to {}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, B)$ *is injective, and*
(3) *every deformation of Y comes from deforming $\mathcal{B}$.*

Then every deformation of X comes from deforming $\mathcal{A}$. Moreover

$$\dim_{(X)} \mathrm{Hilb}(\mathbb{P}^n) = \dim_{(Y)} \mathrm{Hilb}(\mathbb{P}^n) + \dim M_{\mathcal{B}}(a_{t+c-2})_0 - 1 - {}_0\mathrm{hom}_R(I_Y, I_{X/Y}).$$

Remark 4.7. If $\mathrm{depth}_{I(Z)} B \geq 3$, then $\mathrm{depth}_{I(Z)} I_{X/Y} \geq 3$ and it follows from (2.12) that

$${}_0\mathrm{Ext}^1_B(I_{X/Y}, I_{X/Y}) \ = \ {}_0\mathrm{Ext}^1_B(I_{X/Y}, B) \ = \ 0.$$

Remark 4.8. Let $\mathrm{GradAlg}(H_2, H_1)$ be the "postulation Hilbert-flag scheme", i.e. the representing object of the functor deforming surjections $(B \to A)$ of graded quotients of R of positive depth at $\mathbf{m}$, or equivalently flags $(X \subset Y)$ of closed subschemes of $\mathbb{P}^n$ with Hilbert functions $H_Y = H_B = H_2$ and $H_X = H_A = H_1$. In [22], Prop. 4 (iii), we use theorems of Laudal on deformations of a category ([30]) to show that the forgetful morphism

$$\mathrm{GradAlg}(H_2, H_1) \longrightarrow \mathrm{GradAlg}(H_1)$$

induced by $(X \subset Y) \longrightarrow (X)$, is smooth and has fiber dimension ${}_0\mathrm{hom}_R(I_Y, I_{X/Y})$ at $(X \subset Y)$ provided B is unobstructed and (2) of Theorem 4.6 holds. By (1.1) this conclusion holds for the corresponding forgetful map from the Hilbert-flag scheme into the usual Hilbert scheme provided X and Y are ACM and $\dim X \geq 1$.

Proof. Let $R_T \to A_T$ be any graded deformation of $R \to A$ to a local ring T. By the smoothness of the forgetful map of Remark 4.8, there is a graded deformation $R_T \to B_T$ of $R \to B$ and a morphism $B_T \to A_T$. By the assumption (3) there exists a matrix $\mathcal{B}_T$ such that $B_T = R_T/I_t(\mathcal{B}_T)$. By Lemma 4.2 $\mathcal{B}_T$ also defines a deformation $M_{\mathcal{B}_T}$ of $M_{\mathcal{B}}$.

We will prove that there is a matrix $\mathcal{A}_T$ such that $A_T = R_T/I_t(\mathcal{A}_T)$ and such that we get $\mathcal{B}_T$ by deleting the last column of $\mathcal{A}_T$. Looking at (2.5) and the text before and after (2.5), we see that if we can find a section $B_T \to M_{\mathcal{B}_T}(a_{t+c-2})$ which reduces to $B \to M_{\mathcal{B}}(a_{t+c-2})$ via $(-) \otimes_T k$, we can use this section to define a column v_T which allows us to put $\mathcal{A}_T := [\mathcal{B}_T, v_T]$. Now since we have a deformation $B_T \to A_T$ of $B \to A$, it follows that $I_{X_T/Y_T} := \ker(B_T \to A_T)$ is a deformation of $I_{X/Y} \simeq M_{\mathcal{B}}(a_{t+c-2})^*$. If we sheafify and restrict to U we get an isomorphism $\widetilde{I}_{X/Y}|_U \simeq \widetilde{M}_{\mathcal{B}}(a_{t+c-2})^*|_U$ of invertible sheaves. Hence the flat sheaves $(\widetilde{I}_{X_T/Y_T})^*$ and $\widetilde{M}_{\mathcal{B}_T}(a_{t+c-2})$ are also isomorphic on the set U_T which corresponds to U. Taking global sections, $H^0_*(U_T,-)$, of $\widetilde{B}_T \to (\widetilde{I}_{X_T/Y_T})^*$, we get a map which fits into a commutative diagram

$$\begin{array}{ccccc} B_T & \longrightarrow & H^0_*(U_T, \widetilde{M}_{\mathcal{B}_T}(a_{t+c-2})) & \simeq & M_{\mathcal{B}_T}(a_{t+c-2}) \\ \downarrow & & \downarrow & & \\ B & \longrightarrow & H^0_*(U, \widetilde{M}_{\mathcal{B}}(a_{t+c-2})) & \simeq & M_{\mathcal{B}}(a_{t+c-2}) \end{array}$$

where the lower isomorphism follows from the fact that $M_{\mathcal{B}}$ is maximally CM, i.e. from

$$\operatorname{depth}_{I(Z)} M_{\mathcal{B}} = \operatorname{depth}_{I(Z)} B \geq 2 \ .$$

Note that since an $M_{\mathcal{B}}$-regular sequence lifts to an $M_{\mathcal{B}_T}$-regular sequence, we also have sufficient depth to get the upper isomorphism. Hence we get a section $B_T \to M_{\mathcal{B}_T}(a_{t+c-2})$ and an induced matrix $\mathcal{A}_T$, as required.

Let $A' = R_T/I_t(\mathcal{A}_T)$. We *claim* that $A' = A_T$, i.e. that $I'_T = I_{X_T/Y_T}$ where $I'_T = \ker(B_T \to A')$. Let $T_r := T/\mathfrak{m}_T^r$, $B_{T_r} := B_T \otimes_T T_r$,

$$I_{X_{T_r}/Y_{T_r}} := \ker(B_{T_r} \to A_T \otimes_T T_r) \ , \quad I'_{T_r} = \ker(B_{T_r} \to A' \otimes_T T_r)$$

and $S := T_{r-1}$. To prove the claim we first show that $I'_{T_r} = I_{X_{T_r}/Y_{T_r}}$ for every integer $r > 0$. To see that this follows from the assumption (1), we suppose by induction that $I'_S = I_{X_S/Y_S}$ as ideals of B_S. Then I'_{T_r} and $I_{X_{T_r}/Y_{T_r}}$ are two deformations of the same ideal $I_{X_S/Y_S} \hookrightarrow B_S$ to T_r and their difference, as *graded* B_{T_r} *modules*, corresponds to an element,

$$\text{diff} \in {}_0\operatorname{Ext}^1_B(I_{X/Y}, I_{X/Y}) \otimes_k (\mathfrak{m}_T^{r-1} \cdot T_r)$$

which via ρ^1 maps to a difference, $o_1 - o_2 \in {}_0\operatorname{Ext}^1_B(I_{X/Y}, B) \otimes_k (\mathfrak{m}_T^{r-1} \cdot T_r)$ where o_i are the following obstructions. One of them, say o_1 (resp. the other o_2) is the obstruction of deforming the *graded morphism* $I'_S \hookrightarrow B_S$ (i.e. the ideal) to a graded morphism between I'_T and B_T (resp. between $I_{X_{T_r}/Y_{T_r}}$ and B_{T_r}), cf. [22], Rem. 3 for a similar situation. Since I'_T and $I_{X_{T_r}/Y_{T_r}}$ are *ideals* in B_{T_r}, such graded morphisms exist. Hence $o_i = 0$ for $i = 1, 2$, whence diff $= 0$ by the injectivity of ρ^1, and we conclude that $I'_{T_r} = I_{X_{T_r}/Y_{T_r}}$.

To get the claim let $A'' := B_T/(I'_T + I_{X_T/Y_T})$. It suffices to show that the natural maps $A' \to A''$ and $A_T \to A''$ are isomorphisms. Note that $A'' \otimes_T k \simeq A'' \otimes_{R_T} R \simeq A$ and that we have similar isomorphisms for A' and A_T. Notice also that every maximal ideal of R_T lies over $\mathfrak{m}_T$. Hence we get both isomorphisms by the lemma of Nakayama, Azumaya and Krull ([31], Lem. 1.M), i.e. by "Nakayama's lemma" if we can show that A'' is T-flat. But by the proof in the paragraph above the induced maps $A' \otimes_T T_r \to A'' \otimes_T T_r$ are isomorphism for every $r > 0$. It follows that $A'' \otimes_T T_r$ is T_r-flat since $A' \otimes_T T_r$ is! Since A'' is idealwise separated for $\mathfrak{m}_T$, we get that A'' is T-flat by Bourbaki's local criterion of flatness (see Thm. 20.C of [31]).

It remains to prove the dimension formula. Recall that there is a "standard" diagram (whose square is cartesian)

(4.2)

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & {}_0\mathrm{Hom}_R(I_Y, I_{X/Y}) & & \\ & & & & \downarrow & & \\ & & A^1 & \xrightarrow{T_{pr_1}} & {}_0\mathrm{Hom}_R(I_Y, B) & & \\ & & \downarrow & \square & \downarrow & & \\ {}_0\mathrm{Hom}(I_{X/Y}, A) & \hookrightarrow & {}_0\mathrm{Hom}(I_X, A) & \longrightarrow & {}_0\mathrm{Hom}_R(I_Y, A) & \xrightarrow{\delta} & {}_0\mathrm{Ext}^1_B(I_{X/Y}, A) \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

which defines the tangent space A^1 of the Hilbert flag scheme $\mathrm{GradAlg}(H_2, H_1)$ at $(B \to A)$ and where the morphisms T_{pr_1} and δ are natural maps (cf. [22], (10) and note that the algebra cohomology group ${}_0\mathrm{H}^2(B, A, A) \simeq {}_0\mathrm{Ext}^1_B(I_{X/Y}, A)$ (cf. [22], §1.1)). Under the assumption (2) the vertical sequence is exact. We *claim* that $\delta = 0$. To see it, it suffices to prove that T_{pr_1} is surjective. The cartesian diagram is, however, well understood in terms of the deformation theory of the Hilbert flag scheme. Indeed if we take an arbitrary deformation B_S of B to the dual numbers $S := k[t]/(t^2)$, then T_{pr_1} is surjective provided we can prove that there is a deformation $(B_S \to A_S)$ of $(B \to A)$ to S. The latter follows from the first part of the proof, or simply, from the assumption (3) because we by (3) get $B_S = R_S/I_t(\mathcal{B}_S)$ for some matrix $\mathcal{B}_S$ and we can take $\mathcal{A}_S = [\mathcal{B}_S, v_S]$ where v_S is any lifting of the last column of $\mathcal{A}$ to S. Letting $A_S := R_S/I_t(\mathcal{A}_S)$ we get the claim.

Since we have $\dim_{(X)} \mathrm{Hilb}(\mathbb{P}^n) = {}_0\mathrm{hom}(I_X, A)$ and $\dim_{(Y)} \mathrm{Hilb}(\mathbb{P}^n) = {}_0\mathrm{hom}(I_Y, B)$ by Lemma 4.4, we get the dimension formula from the big diagram in which $\delta = 0$ provided we can prove that ${}_0\mathrm{hom}(I_{X/Y}, A) = \dim M_{\mathcal{B}}(a_{t+c-2})_0 - 1$. To see it we apply $\mathrm{Hom}(I_{X/Y}, -)$ onto $0 \to I_{X/Y} \to B \to A \to 0$. If we use that $\mathrm{Hom}(I_{X/Y}, B) \simeq M_{\mathcal{B}}(a_{t+c-2})$, see (2.11), we get the exact sequence

$$(4.3) \quad 0 \to B \to M_{\mathcal{B}}(a_{t+c-2}) \to \mathrm{Hom}(I_{X/Y}, A) \to \mathrm{Ext}^1_B(I_{X/Y}, I_{X/Y}) \to \mathrm{Ext}^1_B(I_{X/Y}, B),$$

and we conclude by the assumption (1). □

Remark 4.9. Suppose $\tau_{X/Y}$ is not injective. Then the vertical sequence in the diagram (4.2) is not exact, and δ may be non-zero. It is, however, easy to enlarge the diagram (4.2) to a diagram of exact horizontal and vertical sequences by including $\ker \tau_{X/Y}$ and $\operatorname{im} \delta$ in the diagram. From this enlarged diagram it follows that

$${}_0\mathrm{hom}(I_X, A) = {}_0\mathrm{hom}(I_{X/Y}, A) + h^0(\mathcal{N}_Y) - {}_0\mathrm{hom}(I_Y, I_{X/Y}) + \dim\ker \tau_{X/Y} - \dim \operatorname{im} \delta$$

since we have $\mathrm{Hom}(I_Y, B) \simeq H^0_*(Y, \mathcal{N}_Y)$ by (2.12) and $\mathrm{depth}_{\mathbf{m}} B \geq 2$. The displayed formula also holds if $\tau_{X/Y}$ is injective.

Theorem 4.10. *Suppose either $c = 2$ and $n \geq 2$, or $3 \leq c \leq 4$, $n - c \geq 2$ and $a_{i-\min(3,t)} \geq b_i$ for $\min(3, t) \leq i \leq t$. If $W(\underline{b}; \underline{a}) \neq \emptyset$, then $\overline{W(\underline{b}; \underline{a})}$ is a generically smooth irreducible component of $\mathrm{Hilb}(\mathbb{P}^n)$ of dimension*

$$\lambda_c + K_3 + ... + K_c\ .$$

Moreover every deformation of a general $(X) \in W(\underline{b};\underline{a})$ comes from deforming $\mathcal{A}$.

For $c > 2$ this result is really [25], Thm. 5.1 and Cor. 5.3 except for the final statement. Since, however, the proof of Theorem 4.10 is to apply Theorem 4.6 inductively to the flag (2.8), starting with the codimension 2 case where we by Lemma 4.11 know that the final statement holds, the proof of [25] extends to get Theorem 4.10 for $c > 2$. If $c = 2$ we get the other conclusions (and even more) from [11] for $n > 2$ and from works of Gotzmann and others for $n = 2$ as explained in [23], Rem. 22 (i), or see [25], Rem. 4.6 for a direct approach to $\dim W(\underline{b};\underline{a}) = \lambda_2$. (We also get all conclusions for $c = 2$ by combining Lemma 4.11 and Lemma 4.4.)

Lemma 4.11. *If $X = \mathrm{Proj}(A)$, $A = R/I_t(\mathcal{A})$, is a standard determinantal scheme of codimension 2 in $\mathbb{P}^n$ and $n \geq 2$, then every deformation of X comes from deforming $\mathcal{A}$.*

Proof. Let $\mathcal{A} = (f_{ij})$ be a homogeneous $t \times (t+1)$ matrix which represents the morphism φ^* of (2.1) and let $R_T \to A_T$ be a graded deformation of $R \to A$ to a local ring T. To see that A_T is of the form $A_T = R_T/I_t(\mathcal{A}_T)$ for some matrix $\mathcal{A}_T$ reducing to $\mathcal{A}$ via $(-) \otimes_T k$, we consider the canonical module $K_A = \mathrm{Ext}^2_R(A,R)(-n-1)$. Note that since $c = 2$ we have $K_A(n+1-\ell_1) = M$, where $G^* \xrightarrow{\varphi^*} F^* \to M \to 0$ is a part of the Buchsbaum-Rim complex, cf. (2.4) and (2.7). Now we observe that $K_{A_T} := \mathrm{Ext}^2_{R_T}(A_T,R_T)(-n-1)$ is a (flat) graded deformation of K_A to T because $\mathrm{Ext}^i_R(A,R) = 0$ for $i \neq 2$ ([8], Proposition (A1)). It follows that $K_{A_T}(n+1-\ell_1) = \mathrm{coker}(\varphi^*_T)$ where φ^*_T corresponds to some matrix $\mathcal{A}_T := (f_{ij,T})$, as in (4.1).

Let $A' := R_T/I_t(\mathcal{A}_T)$. It suffices to show that A' and A_T are isomorphic as R_T quotients. Looking to the Eagon-Northcott complex associated to A' over R_T and dualizing, i.e. applying $\mathrm{Hom}_{R_T}(-,R_T)$ to it, we get back the part of the Buchsbaum-Rim complex where φ^*_T appeared (up to twist). It follows that $K_{A_T} \simeq K_{A'}$ where $K_{A'} := \mathrm{Ext}^2_{R_T}(A',R_T)(-n-1)$. Note that the Buchsbaum-Rim complex above is a free resolution of $K_{A'}(n+1-\ell_1)$ over R_T since it is T-flat and reduces to a R-free resolution via a $(-) \otimes_T k$. Applying $\mathrm{Hom}_{R_T}(-,R_T)$, we get $A' \simeq \mathrm{Ext}^2_{R_T}(K_{A'},R_T)(-n-1)$. Similarly by dualizing twice an R_T-free resolution of A_T we show that $A_T \simeq \mathrm{Ext}^2_{R_T}(K_{A_T},R_T)(-n-1)$ and we are done. ☐

Remark 4.12. If we apply Theorem 4.6 successively to the flag (2.8) it is straightforward to generalize Thm. 5.1 of [25] to the zero dimensional case, i.e. we may replace the condition $n \geq 1$ of [25], Thm. 5.1 by $n \geq 0$ provided we in (i) of Thm. 5.1 (and correspondingly in (ii) of Thm. 5.1) replace the depth ≥ 3 condition by the condition (1) of Theorem 4.6.

There is a variation to Theorem 4.6 that we will use in the case $n = c$ ($\dim X = 0$) in which we mainly replace the injectivity assumption in (1) for the Ext^1-groups with the injectivity assumption for the corresponding Ext^2-groups. More precisely let

$$\rho^i : {}_0\mathrm{Ext}^i_B(I_{X/Y},I_{X/Y}) \to {}_0\mathrm{Ext}^i_B(I_{X/Y},B). \tag{4.4}$$

be the map induced by $I_{X/Y} \hookrightarrow B$. Then we have

Proposition 4.13. *Let $X = \mathrm{Proj}(A) \subset Y = \mathrm{Proj}(B)$ be good determinantal schemes defined by $\mathcal{A}$ and $\mathcal{B}$ where $\mathcal{B}$ is obtained by deleting the last column of $\mathcal{A}$. Let $Z \subset Y$ be a closed subset such that $U := Y - Z \hookrightarrow \mathbb{P}^n$ is a local complete intersection and suppose*

(1) ${}_0\mathrm{Ext}^1_B(I_{X/Y}, A) = 0$ *(i.e.,* ρ^1 *surjective and* ρ^2 *injective) and* $\mathrm{depth}_{I(Z)} B = 2$,
(2) $\tau_{X/Y} : {}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, I_{X/Y}) \hookrightarrow {}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, B)$ *is injective, and*
(3) Y *is unobstructed (this is weaker than* (3) *in Theorem 4.6).*

Then A *is unobstructed and the postulation Hilbert scheme* $\mathrm{Hilb}^{H_A}(\mathbb{P}^n)$ *satisfies*

$$\dim_{(X)} \mathrm{Hilb}^{H_A}(\mathbb{P}^n) = \dim_{(Y)} \mathrm{Hilb}^{p_Y}(\mathbb{P}^n) + \dim M_{\mathcal{B}}(a_{t+c-2})_0 - 1 - {}_0\mathrm{hom}_R(I_Y, I_{X/Y}) + \dim\ker\rho^1.$$

Proof. Let $T \to S$ be a small artinian surjection with kernel $\mathfrak{a}$, and let $R_S \to A_S$ be any graded deformation of $R \to A$ to S. By Remark 4.8, there is a graded deformation $R_S \to B_S$ of $R \to B$ and a morphism $B_S \to A_S$. By assumption (3) and (1.1) there exists a deformation $R_T \to B_T$ of $R_S \to B_S$ to T. It is well known that the algebra cohomology group ${}_0\mathrm{H}^2(B, A, A) \otimes_k \mathfrak{a}$ contains the obstruction of deforming $B_S \to A_S$ further to B_T and that there is an injection ${}_0\mathrm{H}^2(B, A, A) \hookrightarrow {}_0\mathrm{Ext}^1_B(I_{X/Y}, A)$ ([16], exp. VI, [22], §1.1). The rightmost group vanishes by (1), and it follows that A is unobstructed.

Finally we get the dimension formula from the diagram (4.2). Indeed the arguments are almost exactly the same as in the proof of Theorem 4.6 with the variation that (4.3) now implies

$${}_0\mathrm{hom}(I_{X/Y}, A) = \dim M_{\mathcal{B}}(a_{t+c-2})_0 - 1 + \dim\ker\rho^1.$$

□

Remark 4.14. We say that "A is unobstructed along any graded deformation of B" (call this phrase (*)) if for every small artinian surjection $T \to S$ and for *every* graded deformation $B_S \to A_S$ of $B \to A$ to S, there exists, for *every* graded deformation B_T of B_S to T, a graded deformation $B_T \to A_T$ reducing to $B_S \to A_S$ via $(-) \otimes_T S$. It is clear from the proof above that ${}_0\mathrm{Ext}^1_B(I_{X/Y}, A) = 0$ implies (*) and moreover that we can generalize Proposition 4.13 by replacing the assumption ${}_0\mathrm{Ext}^1_B(I_{X/Y}, A) = 0$ by (*).

Now we come to the main results of this paper which are direct consequences of Theorem 4.6 and Proposition 4.13. We start with determinantal curves whose result we will need in the zero dimensional case. Note that the result below is known ([24], Cor. 10.15 and Rem. 10.9 for $c = 3$, [25], Rem. 5.4 and Cor. 5.7 for $4 \le c \le 5$) except for the final statement of (i) and most statements on the codimension in (ii) and (iii). With notations as in Theorem 4.6 we have

Proposition 4.15. *Let* $X = \mathrm{Proj}(A)$, $A = R/I_t(\mathcal{A})$, *be general in* $W(\underline{b}; \underline{a})$ *and suppose* $a_{i-\min(3,t)} \ge b_i$ *for* $\min(3, t) \le i \le t$, $\dim X = n - c = 1$ *and* $3 \le c \le 5$ *(and* $\mathrm{char} k = 0$ *if* $c = 5$*). If* $Y = \mathrm{Proj}(B)$ *is defined by the vanishing of the maximal minors of* $\mathcal{B}$ *where* $\mathcal{B}$ *is obtained by deleting the last column of* $\mathcal{A}$*, then the following statements are true:*

(i) *If* $\tau_{X/Y} : {}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, I_{X/Y}) \to {}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, B)$ *is injective, then* X *is unobstructed and* $\overline{W(\underline{b}; \underline{a})}$ *is a generically smooth irreducible component of* $\mathrm{Hilb}^p(\mathbb{P}^n)$ *of dimension* $\lambda_c + K_3 + ... + K_c$*. Moreover every deformation of* X *comes from deforming* $\mathcal{A}$.

(ii) *If* ${}_0\mathrm{Ext}^1_A(I_X/I_X^2, A) = 0$*, then* X *is unobstructed,* $\dim W(\underline{b}; \underline{a}) = \lambda_c + K_3 + ... + K_c$ *and*

$$\mathrm{codim}_{\mathrm{Hilb}^p(\mathbb{P}^n)} \overline{W(\underline{b}; \underline{a})} = \dim\ker\tau_{X/Y} - {}_0\mathrm{ext}^1_B(I_{X/Y}, A)\ .$$

(iii) *We always have* $\dim W(\underline{b}; \underline{a}) = \lambda_c + K_3 + ... + K_c$ *and*

$$\mathrm{codim}_{\mathrm{Hilb}^p(\mathbb{P}^n)} \overline{W(\underline{b}; \underline{a})} \le \dim\ker\tau_{X/Y}\ .$$

Moreover if ${}_0\mathrm{Ext}^1_B(I_{X/Y}, A) = 0$*, then we have equality in the codimension formula if and only if* X *is unobstructed.*

Proof. In all cases we use Theorem 3.5 to get $\dim W(\underline{b}; \underline{a}) = \lambda_c + K_3 + ... + K_c$.

(i) Since Theorem 4.10 applies to $W(\underline{b}; \underline{a'}) \ni (Y)$ where $\underline{a'} = a_0, a_1, ..., a_{t+c-3}$, we get (i) from Theorem 4.6, Remark 2.2 and Lemma 4.4.

(ii) The vanishing of the Ext^1-group of (ii) implies that X is unobstructed (X is l.c.i. by Remark 2.2), and moreover that $\mathrm{im}\,\delta \simeq {}_0\mathrm{Ext}^1_B(I_{X/Y}, A)$, cf. the diagram (4.2) and continue the horizontal exact sequence as a long exact sequence of algebra cohomology. Since we have $h^0(\mathcal{N}_Y) - \dim W(\underline{b}; \underline{a'}) = 0$ by Theorem 4.10 and ${}_0\hom(I_{X/Y}, A) = \dim M_{\mathcal{B}}(a_{t+c-2})_0 - 1$ by (4.3) and Remark 4.7, we conclude by Remark 4.9 and the final dimension formula of Theorem 3.5.

(iii) As in (ii) we get ${}_0\hom(I_X, A) - \dim W(\underline{b}; \underline{a}) = \dim\ker \tau_{X/Y} - \mathrm{im}\,\delta$ and hence the inequality of (iii). If the ${}_0\mathrm{Ext}^1_B(I_{X/Y}, A)$ vanishes, then $\mathrm{im}\,\delta = 0$, and since one knows that X is unobstructed if and only if we have equality in $h^0(\mathcal{N}_X) \geq \dim_{(X)} \mathrm{Hilb}^p(\mathbb{P}^n)$, we conclude easily. □

Remark 4.16 (for the case where the codimension of X in $\mathbb{P}^n$ is 3, i.e. $c = 3$)**.**

(i) Since Y is licci ([24], Def. 2.10) for $c = 3$, we always have ${}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, B) = 0$ by [2] (or see [18] or [24], Prop. 6.17) and hence we get $\ker \tau_{X/Y} \simeq {}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, I_{X/Y})$.

(ii) It is shown in [24], Cor. 10.11 (for $n - c = 1$) and Cor. 10.17 (for $n - c = 0$) that ${}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, I_{X/Y}) = 0$ provided $a_{t+1} > a_t + a_{t-1} - b_1$. Indeed the proofs of [24] (or [25], Cor. 5.10 (i)) show ${}_0\mathrm{Ext}^1_R(I_Y, I_{X/Y}) = 0$ by mainly using the R-free minimal resolution of I_Y and the degree of the minimal generators of $I_{X/Y}$ which we get from (2.3). Hence we can conclude by the injection ${}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, I_{X/Y}) \hookrightarrow {}_0\mathrm{Ext}^1_R(I_Y, I_{X/Y})$. The vanishing of ${}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, I_{X/Y})$ is, however, much more common than given by the above argument. Indeed examining many examples by Macaulay 2 ([13]) in the range $a_0 > b_t = b_1$ we almost always got ${}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, I_{X/Y}) = 0$ provided $a_{t+1} > 3 + b_t - n + c$ and $0 \leq n - c \leq 1$.

Example 4.17 (determinantal curves in $\mathbb{P}^4$, i.e. with $c = 3$)**.** (i) Let $\mathcal{B}$ be a general 2×3 matrix of linear entries and let $\mathcal{A} = [\mathcal{B}, v]$ where the coordinates of the column v are general polynomials of the same degree m, $m > 0$. The vanishing of all 2×2 minors defines a smooth curve $X = X_m$ of degree $3m + 1$ and genus $3m(m - 1)/2$ in $\mathbb{P}^4$. By Macaulay 2 (mainly),

$$ {}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, I_{X_m/Y}) = 0 \ \text{ if and only if } \quad m \neq 2. $$

Its dimension is 1 if $m = 2$ in which case ${}_0\mathrm{Ext}^1_A(I_{X_m}/I^2_{X_m}, A) = 0$ and ${}_0\mathrm{Ext}^1_B(I_{X_m/Y}, A) = 0$. Note that we above only need to use Macaulay 2 for $m \leq 2$ because the condition $a_{t+1} > a_t + a_{t-1} - b_1$ of Remark 4.16 (ii) is equivalent to $m > 2$. It follows from Proposition 4.15 (i) that $\overline{W(\underline{b}; \underline{a})}$ is a generically smooth irreducible component of $\mathrm{Hilb}^p(\mathbb{P}^4)$ of dimension $\lambda_3 + K_3$ for $m \neq 2$, and from either (ii) or (iii) that X_m is unobstructed and $\mathrm{codim}_{\mathrm{Hilb}(\mathbb{P}^4)} \overline{W(\underline{b}; \underline{a})} = {}_0\mathrm{ext}^1_B(I_Y/I_Y^2, I_{X_m/Y}) = 1$ for $m = 2$. Hence if $m = 2$, then $\dim_{(X_m)} \mathrm{Hilb}(\mathbb{P}^4) = \lambda_3 + K_3 + 1$. This coincides with the $c = 3$ case of Example 4.1!

Finally computing λ_3 and K_3 by their definitions, we get $\lambda_3 + K_3 = 17 + (m+1)(3m+4)/2$ for $m > 1$ and 21 for $m = 1$.

(ii) Let $\mathcal{A}$ be a general 2×4 matrix whose columns consist of general polynomials of the same degree, 1, 2, 3 and m, $m \geq 3$ respectively. Put $\mathcal{A} = [\mathcal{B}, v]$ where the coordinates of the column v are all of degree m. The vanishing of all 2×2 minors of $\mathcal{A}$ defines a smooth curve $X =: X_m$ of degree $11m + 6$ and genus $(11m^2 + 29m + 8)/2$ in $\mathbb{P}^4$. By Remark 4.16 we get ${}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, I_{X_m/Y}) = 0$ for $m > 5$, but a Macaulay 2 computation shows this vanishing also for $3 \leq m \leq 5$. It follows from Proposition 4.15 (i) that $\overline{W(\underline{b};\underline{a})}$ is a generically smooth irreducible component of $\mathrm{Hilb}^p(\mathbb{P}^4)$ of dimension $\lambda_3 + K_3 =$

$$85 + m(11m - 5)/2 \quad \text{for } m > 3 \ , \quad \text{and} \quad 126 \ \text{ for } m = 3.$$

We can also analyze the cases $m = 1$ and 2 by using Proposition 4.15 (i). Note that we now delete the column of degree 3 polynomials to define $\mathcal{B}$, i.e. we let Y be defined by the maximal minors of the 2×3 matrix $\mathcal{B}$ consisting of linear (resp. degree 2 and m) entries in the first (resp. second and third) column. If $m = 1$ (resp. $m = 2$) one verifies that ${}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, I_{X_m/Y}) = 0$ by Macaulay 2 and we get that $\overline{W(\underline{b};\underline{a})}$ is a generically smooth irreducible component of $\mathrm{Hilb}^H(\mathbb{P}^3)$ of dimension 66 (resp. 96).

Remark 4.18. We have checked the vanishing of ${}_0\mathrm{Ext}^1_A(I_{X_m}/I^2_{X_m}, A)$ for several $m \geq 1$ in Example 4.17 (ii). It seems that this group is always non-zero for every $m \geq 1$. This, we think, shows that the results presented here are quite strong because it is hard to show unobstructedness and to find $\dim_{(X_m)} \mathrm{Hilb}^H(\mathbb{P}^n)$ when even the "smallest known obstruction group, ${}_0\mathrm{Ext}^1_A(I_{X_m}/I^2_{X_m}, A)$," does not vanish.

Now we consider zero dimensional determinantal schemes ($n - c = 0$). Indeed Theorem 4.6 with $\mathrm{depth}_{I(Z)} B = 2$ and Proposition 4.13 are designed to take care of this case. We restrict our attention to a general X which through Remark 2.2 imply that all depth conditions of Theorem 4.6 and Proposition 4.13 are satisfied. Then our result leads e.g. to the unobstructedness of A where $X = \mathrm{Proj}(A)$. In fact for special choices of X, A may be obstructed [37]. First we consider codimension $c = 3$ determinantal subschemes and schemes with $c \geq 4$ which we may treat similarly.

Theorem 4.19. *Let $X = \mathrm{Proj}(A)$, $A = R/I_t(\mathcal{A})$, be general in $W(\underline{b};\underline{a})$ and let $Y = \mathrm{Proj}(B)$ and $V = \mathrm{Proj}(C)$ be defined by the vanishing of the maximal minors of $\mathcal{B}$ and $\mathcal{C}$ respectively where $\mathcal{B}$ (resp. $\mathcal{C}$) is obtained by deleting the last column of $\mathcal{A}$ (resp. $\mathcal{B}$). Suppose $\dim X = n - c = 0$, $a_{i-3} \geq b_i$ for $\min(3, t) \leq i \leq t$ and suppose that (3.3) holds. Moreover suppose*

$$\text{either} \quad c = 3 \quad \text{or} \quad [\ 4 \leq c \leq 6 \quad \text{and} \quad \ker \tau_{Y/V} = 0\ ],$$

and suppose char$k = 0$ if $c = 6$. Then the following statements are true:

(i) *If both $\rho^1 : {}_0\mathrm{Ext}^1_B(I_{X/Y}, I_{X/Y}) \to {}_0\mathrm{Ext}^1_B(I_{X/Y}, B)$ and $\tau_{X/Y} : {}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, I_{X/Y}) \to {}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, B)$ are injective, then A is unobstructed and $\overline{W(\underline{b};\underline{a})}$ is a generically smooth irreducible component of the postulation Hilbert scheme $\mathrm{Hilb}^H(\mathbb{P}^c)$ of dimension $\lambda_c + K_3 + ... + K_c$. Moreover every deformation of X comes from deforming $\mathcal{A}$.*

(ii) *If ${}_0\mathrm{Ext}^1_B(I_{X/Y}, A) = 0$ and $\ker \tau_{X/Y} = 0$, then $\overline{W(\underline{b};\underline{a})}$ belongs to a unique generically smooth irreducible component Q of $\mathrm{Hilb}^H(\mathbb{P}^c)$ and the codimension of $\overline{W(\underline{b};\underline{a})}$ in*

$\mathrm{Hilb}^H(\mathbb{P}^c)$ *is* $\dim\ker\rho^1$. *Indeed* A *is unobstructed and*

$$\dim Q = \lambda_c + K_3 + \ldots + K_c + \dim\ker\rho^1.$$

(iii) *If* ${}_0\mathrm{Ext}^1_A(I_X/I_X^2, A) = 0$, *then* A *is unobstructed,* $\dim W(\underline{b};\underline{a}) = \lambda_c + K_3 + \ldots + K_c$ *and*

$$\mathrm{codim}_{\mathrm{Hilb}^H(\mathbb{P}^c)} \overline{W(\underline{b};\underline{a})} = \dim\ker\rho^1 + \dim\ker\tau_{X/Y} - {}_0\mathrm{ext}^1_B(I_{X/Y}, A).$$

(iv) *We always have* $\mathrm{codim}_{\mathrm{Hilb}^H(\mathbb{P}^c)} \overline{W(\underline{b};\underline{a})} \le \dim\ker\rho^1 + \dim\ker\tau_{X/Y}$. *Suppose* ${}_0\mathrm{Ext}^1_B(I_{X/Y}, A) = 0$. *Then we have*

$$\mathrm{codim}_{\mathrm{Hilb}^H(\mathbb{P}^c)} \overline{W(\underline{b};\underline{a})} = \dim\ker\rho^1 + \dim\ker\tau_{X/Y}$$

if and only if A *is unobstructed.*

Proof. In all cases we use Proposition 3.4 to get $\dim W(\underline{b};\underline{a}) = \lambda_c + K_3 + \ldots + K_c$ since the Conjectures hold for $W(\underline{b};\underline{a'}) \ni (Y)$ where $\underline{a'} = a_0, a_1, \ldots, a_{t+c-3}$ by Theorem 3.5. Moreover if we apply Proposition 4.15 (i) to $Y \subset V \subset \mathbb{P}^n$, $(Y) \in W(\underline{b};\underline{a'})$ (provided $c > 3$, if $c = 3$ we apply Theorem 4.10 to $W(\underline{b};\underline{a'}) \ni (Y)$), it follows that *every deformation of* Y *comes from deforming* $\mathcal{B}$.

(i) Using the above statements we easily conclude by Theorem 4.6 and Lemma 4.4.

(ii) Now we use Proposition 4.13 instead of Theorem 4.6. Comparing the dimension formula of Proposition 4.13 with the final one of Proposition 3.4 and using that $\dim_{(Y)} \mathrm{Hilb}^p(\mathbb{P}^c) = \dim W(\underline{b};\underline{a'})$ by Lemma 4.4, we get all conclusions of (ii).

(iii) The vanishing of the Ext^1-group implies that A is unobstructed and that $\mathrm{im}\,\delta \simeq {}_0\mathrm{Ext}^1_B(I_{X/Y}, A)$, cf. (4.2). We have ${}_0\mathrm{hom}(I_{X/Y}, A) = \dim M_{\mathcal{B}}(a_{t+c-2})_0 - 1 + \dim\ker\rho^1$ by (4.3) and $h^0(\mathcal{N}_Y) - \dim W(\underline{b};\underline{a'}) = 0$ by Lemma 4.4. We conclude by Remark 4.9 and the final dimension formula of Proposition 3.4.

(iv) The proof is similar to the last part of (iii), cf. the proof of Proposition 4.15 (iii). □

Remark 4.20. (i) Looking to the proofs we see that we don't need to suppose (3.3) to get the conclusions of (i) and (ii) which don't involve dimension and codimension formulas.

(ii) Note the overlap in (ii) and (iv) of the theorem.

In [24], Ex. 10.18 we considered the example $\mathcal{A} = [\mathcal{B}, v]$ where $\mathcal{B}$ was a general 2×3 matrix of linear entries and where the coordinates of the column v are general polynomials of the same degree m, $m > 2$. Using Macaulay 2 one may easily check that ${}_0\mathrm{Ext}^1_B(I_{X/Y}, I_{X/Y}) = 0$ for $m = 3$. Since $I_{X/Y} = K_B^*(-m-1)$ by (2.6) and (2.7), it follows that ${}_0\mathrm{Ext}^1_B(I_{X/Y}, I_{X/Y})$ is independent of m, and hence vanishes for every $m \ge 3$. The families of zero dimensional schemes given in [24], Ex. 10.18 are therefore generically smooth of known dimension by Theorem 4.19 (i). More advanced examples are given in the examples below where several aspects of Theorem 4.19 are used. Note that the condition (3.3) in Theorem 4.19 is taken care of by Remark 3.6.

Example 4.21 (determinantal zero-schemes in $\mathbb{P}^3$, using mainly Theorem 4.19 (i))**.**

Let $\mathcal{A} = [\mathcal{B}, v]$ be a general 2×4 matrix with linear (resp. cubic) entries in the first and second (resp. third) column and where the entries of v are polynomials of the same degree m, $m \ge 3$. The vanishing of all 2×2 minors defines a reduced scheme $X =: X_m$

of $7m+3$ points in $\mathbb{P}^3$. For $m=3$ one verifies that

$$ {}_0\mathrm{Ext}^1_B(I_{X_m/Y}, I_{X_m/Y}) = 0 \ \text{ and } \ {}_v\mathrm{Ext}^1_B(I_Y/I_Y^2, I_{X_m/Y}) = 0 \text{ and } v \le 0 $$

by Macaulay 2 mainly and since $I_{X_m/Y} = K_B^*(-m+1)$ by (2.6) and (2.7), we get the same conclusion for every $m \ge 3$. It follows from Theorem 4.19 (i) that $\overline{W(\underline{b};\underline{a})}$ is a generically smooth irreducible component of $\mathrm{Hilb}^H(\mathbb{P}^3)$ of dimension $\lambda_3 + K_3 = 7m+25$ (resp. 45) for $m > 3$ (resp. $m=3$).

If $m=1$ or 2 we delete the column of degree 3 polynomials to define $\mathcal{B}$, i.e. we let Y be defined by the maximal minors of the 2×3 matrix $\mathcal{B}$ consisting of linear (resp. degree m) entries in the first and second (resp. third) column. If $m=1$ one verifies (by Macaulay 2) that

$$ {}_0\mathrm{Ext}^1_B(I_{X_m/Y}, I_{X_m/Y}) = {}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, I_{X_m/Y}) = 0 $$

and we get by Theorem 4.19 (i) that $\overline{W(\underline{b};\underline{a})}$ is a generically smooth irreducible component of $\mathrm{Hilb}^H(\mathbb{P}^3)$ of dimension $\lambda_3+K_3 = 22$. If $m=2$ one verifies that ${}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, I_{X_m/Y}) = 0$ and that ${}_0\mathrm{ext}^1_B(I_{X_m/Y}, I_{X_m/Y}) = 2$. Hence we can not use Theorem 4.19 (i), but we can use Theorem 4.19 (ii)! Such cases are more thoroughly explained in the next example. We verify that ${}_0\mathrm{Ext}^1_B(I_{X_m/Y}, B) = 0$, to get $\dim\ker\rho^1 = {}_0\mathrm{ext}^1_B(I_{X_m/Y}, I_{X_m/Y})$, and that ${}_0\mathrm{Ext}^1_B(I_{X_m/Y}, A) = 0$. We conclude that $\overline{W(\underline{b};\underline{a})}$ is contained in a generically smooth irreducible component of $\mathrm{Hilb}^H(\mathbb{P}^3)$ of dimension $\lambda_3 + K_3 + {}_0\mathrm{ext}^1_B(I_{X_m/Y}, I_{X_m/Y}) = 37$.

Example 4.22 (determinantal zero-schemes in $\mathbb{P}^3$, using mainly Theorem 4.19 (ii))**.**

Similar to Example 4.17 (ii) we let $\mathcal{A} = [\mathcal{B}, v]$ be a general 2×4 matrix whose columns consist of general polynomials of the same degree, 1, 2, 3 and m, $m \ge 3$ respectively. The vanishing of all 2×2 minors of $\mathcal{A}$ defines a reduced scheme $X =: X_m$ of $11m+6$ points in $\mathbb{P}^3$. This time Macaulay 2 computations show ${}_0\mathrm{ext}^1_B(I_{X_m/Y}, I_{X_m/Y}) = 2$ and ${}_v\mathrm{Ext}^1_B(I_{X_m/Y}, B) = 0$ for every $m \ge 3$ and every $v \ge 0$ (we only need to check it for $m=3$ because $I_{X_m/Y} = K_B^*(-m+2)$ by (2.6) and (2.7)). It follows that

$$ \dim\ker\rho^1 = {}_0\mathrm{ext}^1_B(I_{X_m/Y}, I_{X_m/Y}) = 2 $$

for every $m \ge 3$, i.e. we can not use Theorem 4.19 (i) at all. We have, however, ${}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, I_{X_m/Y}) = 0$ for $m > 5$ by Remark 4.16 (ii) and ${}_0\mathrm{ext}^1_B(I_Y/I_Y^2, I_{X_m/Y}) = 2$ (resp. 0) for $m=3$ (resp. $3 < m \le 5$) by Macaulay 2. Since ${}_0\mathrm{Ext}^2_B(I_{X_m/Y}, I_{X_m/Y}) = 0$ for $m=3$ and hence for every $m \ge 3$, we get ${}_0\mathrm{Ext}^1_B(I_{X_m/Y}, A) = 0$ for $m \ge 3$. We can therefore apply Theorem 4.19 (ii) in this situation except when $m=3$. In the latter case ${}_0\mathrm{Ext}^1_A(I_{X_m}/I_{X_m}^2, A) = 0$ and Theorem 4.19 (iii) applies. Hence Theorem 4.19 applies for every $m \ge 3$, and we get that $\overline{W(\underline{b};\underline{a})}$ belongs to a unique generically smooth irreducible component of $\mathrm{Hilb}^H(\mathbb{P}^3)$ of codimension 2 (resp. 4) for $m > 3$ (resp. $m=3$). Indeed A is unobstructed and

$$ \dim\overline{W(\underline{b};\underline{a})} = \lambda_3 + K_3 = 11m+35 \quad \text{for } m > 3 \quad \text{and} \quad 67 \ \text{ for } m = 3. $$

We remark that we have checked a possible vanishing of ${}_0\mathrm{Ext}^1_A(I_{X_m}/I_{X_m}^2, A)$ for several $m \ge 3$, and in the range $3 < m \le 6$ this group is non-zero.

Finally to be complete we consider the cases $m=1$ and $m=2$ in which case we will delete the column of degree 3 polynomials to define $\mathcal{B}$ and hence Y. If $m=1$ we get by Macaulay 2 $\dim\ker\rho^1 = {}_0\mathrm{ext}^1_B(I_{X_m/Y}, I_{X_m/Y}) = 2$, ${}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, I_{X_m/Y}) = 0$ and ${}_0\mathrm{Ext}^1_B(I_{X_m/Y}, A) = 0$. We have $\dim \overline{W(\underline{b};\underline{a})} = \lambda_3 + K_3 = 35$ and

$$\operatorname{codim}_{\mathrm{Hilb}^H(\mathbb{P}^3)} \overline{W(\underline{b};\underline{a})} = \dim\ker\rho^1 = 2$$

by Theorem 4.19 (ii). Moreover if $m = 2$ we get $\dim\ker\rho^1 = {}_0\mathrm{ext}^1_B(I_{X_m/Y}, I_{X_m/Y}) = 4$, ${}_0\mathrm{ext}^1_B(I_Y/I_Y^2, I_{X_m/Y}) = 1$, ${}_0\mathrm{Ext}^1_B(I_{X_m/Y}, A) = 0$ and ${}_0\mathrm{Ext}^1_A(I_{X_m}/I_{X_m}^2, A) = 0$ by Macaulay 2. By Theorem 4.19 (iii) we find $\dim W(\underline{b};\underline{a}) = \lambda_3 + K_3 = 53$ and

$$\operatorname{codim}_{\mathrm{Hilb}^H(\mathbb{P}^3)} \overline{W(\underline{b};\underline{a})} = \dim\ker\rho^1 + \dim\ker\tau_{X/Y} = 4 + 1 = 5.$$

Example 4.23 (Using Theorem 4.19 (i) with non-vanishing obstruction groups).

We let $\mathcal{A} = [\mathcal{B}, v]$ be a general 2×4 matrix whose columns consist of general polynomials of the same degree, 2, 2, 4 and m, $m \geq 4$ respectively. The vanishing of all 2×2 minors of $\mathcal{A}$ defines a reduced scheme $X =: X_m$ of $20m + 16$ points in $\mathbb{P}^3$. This time Macaulay 2 computations show ${}_0\mathrm{ext}^1_B(I_{X_m/Y}, I_{X_m/Y}) = 0$ and ${}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, I_{X_m/Y}) = 0$ (resp. $= 1$) for every $m > 4$ (resp. $m = 4$). It follows from Theorem 4.19 (i) that $\overline{W(\underline{b};\underline{a})}$ is a generically smooth irreducible component of $\mathrm{Hilb}^H(\mathbb{P}^3)$ of dimension $\lambda_3 + K_3 = 20m + 49$ for $m > 4$. For $m = 4$ we have verified that ${}_0\mathrm{ext}^1_A(I_{X_m}/I_{X_m}^2, A) = 3$ and in this particular case we have not been able to verify whether A is unobstructed or not. But for every $m > 4$, A is unobstructed by Theorem 4.19 (i)! Moreover we have checked a possible vanishing of ${}_0\mathrm{Ext}^1_A(I_{X_m}/I_{X_m}^2, A)$ for many m, and combined with some theoretical arguments (which we don't take here) we can conclude that this group is always non-zero for every $m \geq 4$. Again, we think, this shows that the results presented here are quite strong because it is really hard to show unobstructedness when even the "smallest known obstruction group, ${}_0\mathrm{Ext}^1_A(I_{X_m}/I_{X_m}^2, A)$," does not vanish.

In the final case $4 \leq c \leq 6$ and $\ker\tau_{Y/V} \neq 0$ where a general $X = \mathrm{Proj}(A) \subset Y = \mathrm{Proj}(B) \subset V = \mathrm{Proj}(C)$ is given by deleting columns as above we can not apply Theorem 4.6 to $X \subset Y$ because there is no reason to expect condition (3) of Theorem 4.6 to be true (that condition is closely related to $\ker\tau_{Y/V} = 0$). But we can still use Proposition 4.13 since condition (3) of Proposition 4.13 is weakened to "Y unobstructed". The natural condition for "Y unobstructed" which also give a formula for $h^0(\mathcal{N}_Y) - \dim W(\underline{b};\underline{a'})$ is ${}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, B) = 0$, cf. the proof of Proposition 4.15 (ii). We get

Proposition 4.24. *With notations as above, suppose* $4 \leq c \leq 6$ *(let* $\mathrm{char}\,k = 0$ *if* $c = 6$*),* $\dim X = n - c = 0$*,* $a_{i-3} \geq b_i$ *for* $\min(3,t) \leq i \leq t$ *and suppose that* (3.3) *holds. Then* $\dim W(\underline{b};\underline{a}) = \lambda_c + K_3 + ... + K_c$ *and the following statements are true:*

(i) *If* ${}_0\mathrm{Ext}^1_B(I_{X/Y}, A) = 0$*,* ${}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, I_{X/Y}) = 0$ *and* ${}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, B) = 0$ *then* A *is unobstructed. Moreover* $W(\underline{b};\underline{a})$ *is contained in a unique generically smooth irreducible component of* $\mathrm{Hilb}^H(\mathbb{P}^c)$ *of codimension* $\dim\ker\rho^1 + \dim\ker\tau_{Y/V} - {}_0\mathrm{ext}^1_C(I_{Y/V}, B)$.

(ii) *We always have* $\operatorname{codim}_{\mathrm{Hilb}^H(\mathbb{P}^c)} \overline{W(\underline{b};\underline{a})} \leq \dim\ker\rho^1 + \dim\ker\tau_{X/Y} + \dim\ker\tau_{Y/V}$. *Suppose* ${}_0\mathrm{Ext}^1_B(I_{X/Y}, A) = 0$ *and* ${}_0\mathrm{Ext}^1_C(I_{Y/V}, B) = 0$*. Then we have*

$$\operatorname{codim}_{\mathrm{Hilb}^H(\mathbb{P}^c)} \overline{W(\underline{b};\underline{a})} = \dim\ker\rho^1 + \dim\ker\tau_{X/Y} + \dim\ker\tau_{Y/V}$$

if and only if A *is unobstructed (e.g.* ${}_0\mathrm{Ext}^1_A(I_X/I_X^2, A) = 0$*).*

Proof. We have $\dim W(\underline{b};\underline{a}) = \lambda_c + K_3 + ... + K_c$ by Proposition 3.4 since Theorem 3.5 applies to $W(\underline{b};\underline{a'})$ where $\underline{a'} = a_0, a_1, ..., a_{t+c-3}$.

(i) This follows from Proposition 4.13 and Proposition 4.15 (ii) and by comparing the dimension formula of Proposition 4.13 with the final one of Proposition 3.4.

(ii) Combining Remark 4.9 and the final formula of Proposition 3.4 with (4.3), we get

$${}_0\hom(I_X, A) - \dim W(\underline{b};\underline{a}) = h^0(\mathcal{N}_Y) - \dim W(\underline{b};\underline{a'}) + \dim\ker\rho^1 + \dim\ker\tau_{X/Y} - \operatorname{im}\delta.$$

By the same argument we have $h^0(\mathcal{N}_Y) - \dim W(\underline{b};\underline{a'}) \leq \dim\ker\tau_{Y/V}$ and moreover, if ${}_0\mathrm{Ext}^1_C(I_{Y/V}, B) = 0$, then equality holds. Hence we get the inequality of (ii), and furthermore, if the two Ext^1-groups of (ii) vanish then the inequality of (ii) is an equality if and only if $\dim_{(X)} \mathrm{Hilb}^H(\mathbb{P}^c) = {}_0\hom(I_X, A)$ and we are done. □

Example 4.25 (determinantal zero dimensional schemes in $\mathbb{P}^4$, i.e. with $c = 4$)**.**

Let $\mathcal{A} = [\mathcal{B}, v]$ be a general 2×5 matrix with linear (resp. quadratic) entries in the first, second and third (resp. fourth) column and let both entries of the column v be of degree $m \geq 2$. Keeping the notations of Proposition 4.24, we get that the vanishing of all 2×2 minors defines a reduced scheme X of $7m + 2$ points in $\mathbb{P}^4$. One verifies that $\dim\ker\rho^1 = {}_0\mathrm{ext}^1_B(I_{X/Y}, I_{X/Y}) = 3$, ${}_0\mathrm{Ext}^1_B(I_{X/Y}, A) = 0$ and that (3.3) holds by Remark 3.6. Note that we have $\dim\ker\tau_{Y/V} = 1$ and ${}_0\mathrm{Ext}^1_C(I_{Y/V}, B) = 0$ from Example 4.17 (i).

Suppose $m > 2$. Then ${}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, I_{X/Y}) = {}_0\mathrm{Ext}^1_B(I_Y/I_Y^2, B) = 0$ for every $m > 2$ and it follows from Proposition 4.24 (i) that A is unobstructed and $\dim W(\underline{b};\underline{a}) = \lambda_4 + K_3 + K_4 = 7m + 31$. Hence $W(\underline{b};\underline{a})$ is contained in a unique generically smooth irreducible component of the postulation Hilbert scheme $\mathrm{Hilb}^H(\mathbb{P}^4)$ and,

$$\operatorname{codim}_{\mathrm{Hilb}^H(\mathbb{P}^4)} \overline{W(\underline{b};\underline{a})} = \dim\ker\rho^1 + \dim\ker\tau_{Y/V} = 3 + 1 = 4.$$

Suppose $m = 2$. Since $\dim\ker\tau_{X/Y} = {}_0\mathrm{ext}^1_B(I_Y/I_Y^2, I_{X/Y}) = 4$ and ${}_0\mathrm{Ext}^1_A(I_X/I_X^2, A) = 0$, it follows from Proposition 4.24 (ii) that A is unobstructed and that $\dim W(\underline{b};\underline{a}) = \lambda_4 + K_3 + K_4 = 44$. Hence $W(\underline{b};\underline{a})$ is contained in a unique generically smooth irreducible component of $\mathrm{Hilb}^H(\mathbb{P}^4)$ and,

$$\operatorname{codim}_{\mathrm{Hilb}^H(\mathbb{P}^4)} \overline{W(\underline{b};\underline{a})} = \dim\ker\rho^1 + \dim\ker\tau_{X/Y} + \dim\ker\tau_{Y/V} = 3 + 4 + 1 = 8.$$

In this case we see that all three kernels of Proposition 4.24 (ii) contribute to the codimension of $W(\underline{b};\underline{a})$ in $\mathrm{Hilb}^H(\mathbb{P}^4)$!

Remark 4.26. If we apply Theorem 4.6 successively to the flag (2.8) we get Prop. 10.12 and Thm. 10.13 of [24] in a correct version (the injectivity of ρ^1, i.e. the assumption (1) of Theorem 4.6 in the case $\mathrm{depth}_{I(Z)} B = 2$ lacked in [24]). Indeed in [25], Rem. 6.3 we announced that some results in §10 of [24] were inaccurate, and in the new hypothesis (*) of Rem. 6.3 we increased the depth assumption of the corresponding hypothesis in [24] by 1 to get valid results. The new hypothesis (*) applies to determinantal schemes of positive dimension, i.e. the results of [24], §10 hold in this case. In the zero dimensional case we introduced, in addition to (*) of Rem. 6.3, an assumption (Rem. 6.3 (ii)), which is equivalent to the injectivity of ρ^1. This assumption makes the results of [24], §10 correct in the zero dimensional case. In [25], Rem. 6.3 (ii) we indicate a proof for this claim, and now Theorem 4.6 provides us with another proof. In [25], Rem. 6.3 (i) and (iii), we claimed that e.g. the unobstructedness of A also implied *all* results of [24], §10, but this is a little inaccurate because the very final result of [24] (Cor. 10.17) uses the injectivity of ρ^1 to get the dimension formula. E.g. in Example 4.22 for $m > 3$ (resp. Example 4.21 with $m = 2$) the formula of Cor. 10.17 gives $\dim_{(X)} \operatorname{Hilb}^H(\mathbb{P}^3) = \dim W(\underline{b};\underline{a})$, which should be correct according to Rem. 6.3 (i) (resp. Rem. 6.3 (iii)). The correct dimension is, however, $\dim_{(X)} \operatorname{Hilb}^H(\mathbb{P}^3) = \dim W(\underline{b};\underline{a}) + \dim\ker\rho^1$, $\dim\ker\rho^1 = 2$ in both cases. This observation is a reason for writing this paper, namely to provide detailed proofs in the zero dimensional case for the correction "Rem. 6.3 (ii)" and to present several results related to [25], Rem. 6.3 (i) and (iii) (see Proposition 4.13, Theorem 4.19, Proposition 4.24 where we see that we have to add $\dim\ker\rho^1$ to get valid (co)dimension formulas. Note also the obvious misprint in Rem. 6.3, that I_c should have been I_{c-1}). Thus, letting $D_i = R/I_{X_i}$ and $I_i = I_{X_{i+1}/X_i}$, the following hypothesis makes *all* results of [24], §10, true for good determinantal schemes X with $\dim X \geq 0$;

Given $X \subset \mathbb{P}^n$ a good determinantal scheme of dimension $n-c$, we will assume that there exists a flag $X = X_c \subset X_{c-1} \subset \ldots \subset X_2 \subset \mathbb{P}^n$ such that for each $i < c$, the closed embedding $X_{i+1} \hookrightarrow X_i$ is l.c.i. outside some set Z_i of codimension 2 in X_{i+1} ($\operatorname{depth}_{Z_i} \mathcal{O}_{X_{i+1}} \geq 2$). Moreover, we suppose $X_2 \hookrightarrow \mathbb{P}^n$ is an l.c.i. in codimension ≤ 1 and if $c = n$ we suppose that ${}_0\operatorname{Ext}^1_{D_{c-1}}(I_{c-1}, I_{c-1}) \hookrightarrow {}_0\operatorname{Ext}^1_{D_{c-1}}(I_{c-1}, D_{c-1})$ is injective.

5. Applications to families of Gorenstein quotients

The results of the preceding section lead to many well described generically smooth components of $\operatorname{Hilb}(\mathbb{P}^n)$ of known dimension. Once we have such a component of $\operatorname{Hilb}^{H_A}(\mathbb{P}^n)$ consisting, say, of codimension 3 zero dimensional schemes $X = \operatorname{Proj}(A)$, then regular sections of a twist of the anticanonical module of A lead to artinian Gorenstein codimension 4 quotients through the exact sequence

$$0 \to K_A(-s) \xrightarrow{\sigma} A \to D \to 0 \ . \tag{5.1}$$

Indeed by [22], Thm. 16, we have the following result (true for arbitrary dimension of X).

Theorem 5.1. *Let $A = R/I_A$ be a graded, CM quotient with canonical module K_A, let $X := \operatorname{Proj}(A)$ be locally Gorenstein and let D be the Gorenstein algebra given by a regular section of $\sigma \in (K_A^*)_s$ for some integer s. If $s >> 0$, then D is H_A-generic. Moreover A is unobstructed if and only if D is unobstructed, and*

$$\dim_{(D)} \operatorname{Hilb}^{H_D}(\mathbb{P}^n) = \dim_{(X)} \operatorname{Hilb}^{H_A}(\mathbb{P}^n) + \dim(K_A^*)_s - 1 \ .$$

Here D is said to be H_A-generic if there is an open subset of $\operatorname{Hilb}^{H_D}(\mathbb{P}^n)$ containing (D) whose members D' are quotients of some quotient A' of R with Hilbert function H_A. In fact if X is general in some irreducible component of $\operatorname{Hilb}^{H_A}(\mathbb{P}^n)$ we may suppose D is general in some component of $\operatorname{Hilb}^{H_D}(\mathbb{P}^n)$, see [22], Thm. 24 for the entire correspondence.

To make $s >> 0$ precise, we notice that Theorem 5.1 holds for every integer s satisfying

$${}_s\operatorname{Ext}^1_A(S_2(K_A), K_A) = 0 \ \text{ and } \ {}_{-s}\operatorname{Ext}^i_R(I_A, K_A) = 0 \ \text{ for } i = 0, 1 \ , \tag{5.2}$$

see Lem. 9, Prop. 10 (ii) and proof of Thm. 27 of [23] to understand that we may replace "${}_{-s}\operatorname{Hom}_B(H_2(R, A, A), K_A) = 0$ (e.g. I_A is generically syzygetic) and ${}_{-s}\operatorname{Ext}^1_A(I_A/I_A^2, K_A) =$

0" of [22], Prop. 13 and Thm. 16 with "${}_{-s}\mathrm{Ext}^1_R(I_A,K_A)=0$". Hence we can still use the arguments of [22] which consider minimal resolutions of I_A and K_A to find s. We get

Remark 5.2. (i) If X is a zero-scheme of degree d, we may replace $\dim_{(D)}\mathrm{Hilb}^{H_D}(\mathbb{P}^n)$ by $\dim_{(D)}\mathrm{PGor}(H_D)$, $s >> 0$ by $s \ge 2\,\mathrm{reg}(I_A)$ and $\dim(K_A^*)_s$ by d (cf. [22], Rem. 22).

(ii) If X is an integral curve of degree d and arithmetic genus p, we may replace $s >> 0$ by $s > \max\{2\,\mathrm{reg}(I_A)-2,(4p-4)/d\}$ and $\dim(K_A^*)_s$ by $ds+3-3p$ (cf. [22], Rem. 18).

We finish this paper by taking two far from straightforward examples.

Example 5.3 (Artinian Gorenstein quotients of R of codimension 4)**.**

Take the determinantal zero dimensional scheme X of Example 4.22 in which $\mathcal{A}=[\mathcal{B},v]$ was a general 2×4 matrix whose columns consisted of polynomials of the same degree, 1, 2, 3 and m respectively. In that example $X=\mathrm{Proj}(A)$ was a reduced scheme of $11m+6$ points in $\mathbb{P}^3$ and $\overline{W(\underline{b};\underline{a})}$ was a proper closed subset of a generically smooth component V of $\mathrm{Hilb}^{H_A}(\mathbb{P}^3)$ of codimension 2 (resp. 4 or 5) for $m>3$ and $m=1$ (resp. $m=3$ or 2). Moreover

$$\dim \overline{W(\underline{b};\underline{a})} = \lambda_3 + K_3 = 11m+35 \quad \text{for} \quad m>3$$

and $\dim \overline{W(\underline{b};\underline{a})} =$ 67, 53 or 35 for $m=3,2$ or 1 respectively. Using (2.3) we find the leftmost part of a minimal resolution of A and hence of the general element R/J of V to be $0\to R(-m-6)^3\to \dots$, from which we deduce $\mathrm{reg}(J)=m+4$. Moreover writing the Hilbert function H as $(H(0),H(1),H(2),\dots)$ we have

$$H_{R/J}=H_A=(1,4,10,19,\dots,11m-4,11m+3,11m+6,11m+6,\dots).$$

By Theorem 5.1 and Remark 5.2 (i) we get, for every $m\ge 1$ and $s\ge 2m+8$ a generically smooth component of $\mathrm{PGor}(H_D)$ of dimension $22m+42$ (resp. 109, 85 or 53) provided $m\ge 4$ (resp. $m=3$, 2 or 1) where the Hilbert function H_D is given by the "$(s+1)$-tuple"

$$(1,4,10,\dots,11m+3,11m+6,\dots,11m+6,11m+3\dots,10,4,1,0,0,\dots),$$

e.g. the case $m=2$ and $s=12$ yields

$$(1,4,10,18,25,28,28,28,25,18,10,4,1,0,0,\dots).$$

Example 5.4 (One dimensional Gorenstein quotients of R of codimension 4)**.**

Now we consider the determinantal one dimensional schemes of Example 4.17 in which $\mathcal{A}$ is exactly as in Example 5.3 with the exception that the polynomials are taken from $R:=k[x_0,\dots,x_4]$. In this case $X=\mathrm{Proj}(R/I_A)$ is a smooth curve of degree $d=11m+6$ and genus $p=(11m^2+29m+8)/2$ in $\mathbb{P}^4$, and $\mathrm{reg}(I_A)=m+4$. By Example 4.17 we see that $\overline{W(\underline{b};\underline{a})}$ is a generically smooth irreducible component of $\mathrm{Hilb}^p(\mathbb{P}^4)$ of dimension

$$\lambda_3+K_3 = 85+m(11m-5)/2 \ \ (\text{resp. } 126, 96 \text{ or } 66) \text{ for } m>3 \ (\text{resp. } m=3,2 \text{ or } 1).$$

Since $(4p-4)/d<2m+5$ we apply Theorem 5.1 and Remark 5.2 (ii). Hence for every $m\ge 1$ and $s\ge 2m+7$ we get a generically smooth component V of the postulation Hilbert scheme $\mathrm{Hilb}^{H_D}(\mathbb{P}^4)$ whose general element is a codimension 4 Gorenstein quotient of R and whose h-vector is the $(s+1)$-tuple of Example 5.3. Moreover

$$\dim V = (11m+6)s - 11m^2 - 46m + 75 \ \text{ for } m>3$$

and $\dim V=(11m+6)s-q$ where $q=163,67$ or 4 for $m=3,2$ or 1 respectively.

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